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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, E. C. THOMPSON

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THE GROUPS $\pi_r(V_{n,m})$ (IV)

By G. F. PAECHTER (*Oxford*)

[Received 12 December 1957]

Introduction

THIS is the fourth of a sequence of five papers, the previous ones being (**2**), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those groups which are given in the following tables. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$. A full table of results can be found in (**2**) (I) 249. For the notation used throughout the body of this paper please see (**2**), especially §§ 1, 2, and 3.1. Also please note that sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,5}$.

k	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
1	0	Z_∞	0	Z_∞	Z_{24}
3	Z_2	$Z_4+Z_\infty+Z_\infty$	$Z_2+Z_2+Z_2+Z_2$		
4	$Z_{12}+Z_4+Z_\infty$	$Z_2+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$		
5	Z_2	Z_2+Z_∞	Z_2		
6	Z_{12}	Z_2	$Z_\infty+Z_8$ or Z_4+Z_∞		
$8s-1$	Z_2	Z_8+Z_∞	$Z_2+Z_2+Z_2$		
$8s+3$	Z_2+Z_2	Z_8+Z_∞	$Z_2+Z_2+Z_2$		
$4s+5$	Z_2	Z_∞	Z_2		
$8s$	$Z_{24}+Z_8$	Z_2+Z_2	Z_2		
$8s+4$	Z_4+Z_{48}	Z_2+Z_2	Z_2		
$4s+6$	Z_{12}	Z_2	Z_8		

TABLE FOR $\pi^p_{2,4}$.

$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
0	Z_∞	0	Z_∞	Z_{24}

9. Calculation of $\pi^p_{k,5}$

We consider the fibring $V_{k+5,5}/V_{k+4,4} \to S^{k+4}$ and examine the sequence

$$\text{(D)} \qquad \to \pi_{k+p+1}(S^{k+4}) \xrightarrow{\Delta_*} \pi^p_{k,4} \xrightarrow{i_{k+p*}} \pi^p_{k,5} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+4}) \to.$$

9.1. $k \equiv 3 \pmod 8$.

In this case there is a four-field on S^{k+4} (**1**, **4**), and so the fibring admits a cross-section $\mathfrak{p}$. Hence Theorem 1.1 gives that

$$\pi^p_{k,5} = i_*\pi^p_{k,4} + \mathfrak{p}_*\pi_{k+p}(S^{k+4}).$$

Using the values of $\pi^p_{k,4}$ as calculated in § 8.4, we obtain the values shown in the table for $\pi^p_{k,5}$ when $k \equiv 3 \pmod 8$.

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+5,5}\} = 0 \quad \text{for } k \equiv 3 \pmod 8.$$

9.2. $k \equiv 7 \pmod 8$.

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_2 \to \pi^3_{k,5} \to 0,$$

by § 8.4 (b). But $i^{-1}_{k+3*}(0) \neq 0$ since otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a four-field on S^{k+4}, which is impossible by Theorem 1.1 of (**3**). Also $i^{-1}_{k+3*}(0) = \Delta_* Z_\infty$, and so must be cyclic. Hence

$$\pi^3_{k,5} = Z_2.$$

Note that $\Delta_* \pi_{k+4}(S^{k+4})$ is of order two, whence the image of p_{k+4*} is the Z_∞ subgroup generated by $2\{h_{k+4,k+4}\}$.

To determine the generator of $\pi^3_{k,5}$ we must evaluate $\{t_{k+5,5}\}$ which generates $i^{-1}_{k+3*}(0)$. Consider the sequence associated with the fibring $V_{k+4,4}/S^k \to V_{k+4,3}$, which is of the form

$$\to \pi_{k+3}(S^k) \xrightarrow{i_{k+1,3*}} \pi^3_{k,4} \xrightarrow{p_{k+4,3*}} \pi^2_{k+1,3} \to.$$

Then, by § 2.3 (b), $p_{k+4,3*}\{t_{k+5,5}\} = \{t_{k+5,4}\}$, which is zero by § 8.1. Hence, by exactness,

$$\{t_{k+5,5}\} \in i_{k+1,3*}\, \pi_{k+3}(S^k),$$

and is non-zero by the previous paragraph. Hence, using the result of § 8.4 (b), we have that

$$\{t_{k+5,5}\} = \{i_{k+1,3}\, h_{k,k+3}\},$$

and the generator of $\pi^3_{k,5}$ is $i_{k+2,3*}\, a$, where $p_{k+2,1*}\, a = \{h_{k+1,k+3}\}$.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_8 \to \pi^4_{k,5} \to Z_\infty \to 0,$$

by § 8.4 (d), and since the image of p_{k+4*} is a Z_∞ subgroup by (a). Also $i^{-1}_{k+4*}(0) = t_{k+5,5*}\, \pi_{k+4}(S^{k+3})$, which is generated by

$$h^*_{k+3,k+4}\{t_{k+5,5}\} = h^*_{k+3,k+4}\, i_{k+1,3*}\{h_{k,k+3}\} = i_{k+1,3*}\, h^*_{k+3,k+4}\{h_{k,k+3}\}$$
$$\in i_{k+1,3*}\, \pi_{k+4}(S^k) = 0 \quad \text{since } k \geqslant 7.$$

Hence $i_{k+4*}^{-1}(0) = 0$, whence

$$\pi_{k,5}^4 = Z_8 + Z_\infty,$$

generated by $i_{k+4,1*}a$, where $p_{k+4,1*}a = \{h_{k+3,k+4}\}$, and b such that $p_{k+5,1*}b = 2\{h_{k+4,k+4}\}$. Note that Δ_* is trivial, whence p_{k+5*} is onto.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^5 \xrightarrow{i_{k+5*}} \pi_{k,5}^5 \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 \to \pi_{k,5}^5 \to Z_2 \to 0,$$

by § 8.4 (*f*), and since p_{k+5*} is onto by (*b*). But, by Theorem 4.2 (*b*),

$$\begin{aligned}\pi_{k,5}^5 &\approx \pi_{k+1,4}^4 \\ &= Z_2 + Z_2 + Z_2\end{aligned}$$

by § 8.1. Hence $i_{k+5*}^{-1}(0) = 0$, whence Δ_* is trivial and p_{k+6*} onto, and

$$\pi_{k,5}^5 = Z_2 + Z_2 + Z_2,$$

generated by $\quad i_{k+3,2*}p_{k+3,1*}^{-1}\{h_{k+2,k+5}\}, \quad i_{k+4,1*}a, \quad$ and b,
where $p_{k+4,1*}a = \{h_{k+3,k+5}\}$, and $p_{k+5,1*}b = \{h_{k+4,k+5}\}$.

9.3. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^3 \xrightarrow{i_{k+3*}} \pi_{k,5}^3 \to \pi_{k+3}(S^{k+4}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_2 \to \pi_{k,5}^3 \to 0,$$

by § 8.3 (*b*). As in § 9.2 (*a*) above, $i_{k+3*}^{-1}(0) \neq 0$ since otherwise the fibring $V_{k+5,5}/V_{k+4,4} \to S^{k+4}$ would admit a cross-section by Theorem 1.2, which would imply a four-field on S^{k+4}, which is impossible by Theorem 1.1 in (**3**). Also $i_{k+3*}^{-1}(0) = \Delta_* Z_\infty$, and so must be cyclic. Hence

$$\pi_{k,5}^3 = Z_2.$$

Note also that $\Delta_* \pi_{k+4}(S^{k+4})$ is of order two, whence the image of p_{k+4*} is the Z_∞ subgroup generated by $2\{h_{k+4,k+4}\}$.

To determine the generator of $\pi_{k,5}^3$ we must evaluate $\{t_{k+5,5}\}$ which generates $i_{k+3*}^{-1}(0)$. Consider the sequence associated with the fibring $V_{k+4,4}/S^k \to V_{k+4,3}$, which is of the form

$$\to \pi_{k+3}(S^k) \xrightarrow{i_{k+1,3*}} \pi_{k,4}^3 \xrightarrow{p_{k+4,3*}} \pi_{k+1,3}^2 \to.$$

Then, by § 2.3 (*b*),

$$\begin{aligned}p_{k+4,3*}\{t_{k+5,5}\} &= \{t_{k+5,4}\} \\ &= \{i_{k+3,1}\mathfrak{p}h_{k+2,k+3}\}, \quad \text{by 8.2}\,(a), \\ &\neq 0.\end{aligned}$$

Hence
$$\{t_{k+5,5}\} \notin i_{k+1,3*}\pi_{k+3}(S^k),$$

whence, using the result of § 8.3 (b) we have that

$$\{t_{k+5,5}\} = \{i_{k+3,1}\,\mathfrak{p}h_{k+2,k+3}\} + \lambda\{i_{k+1,3}\,h_{k,k+3}\}, \quad \textit{where } \lambda = 0 \textit{ or } 1.$$

Thus the generator of $\pi^3_{k,5}$ is $\{i_{k+1,4}\,h_{k,k+3}\}$.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \rightarrow,$$

i.e. $$\rightarrow Z_2 \rightarrow Z_2 \rightarrow \pi^4_{k,5} \rightarrow Z_\infty \rightarrow 0 \quad (k > 5)$$

and $$\rightarrow Z_2 \rightarrow Z_2 + Z_2 \rightarrow \pi^4_{5,5} \rightarrow Z_\infty \rightarrow 0 \quad (k = 5)$$

by § 8.3 (c), and since the image of p_{k+4*} is a Z_∞ subgroup by (a). Further, $i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3})$, which is generated by

$$\begin{aligned} h^*_{k+3,k+4}\{t_{k+5,5}\} &= h^*_{k+3,k+4}(i_{k+3,1*}\,\mathfrak{p}_*\{h_{k+2,k+3}\} + \lambda i_{k+1,3*}\{h_{k,k+3}\}) \\ &= i_{k+3,1*}\,\mathfrak{p}_*\,h^*_{k+3,k+4}\{h_{k+2,k+3}\} + \lambda i_{k+1,3*}\,h^*_{k+3,k+4}\{h_{k,k+3}\} \\ &= \begin{cases} i_{k+3,1*}\,\mathfrak{p}_*\{h_{k+2,k+4}\} & (k \geqslant 9) \\ i_{8,1*}\,\mathfrak{p}_*\{h_{7,9}\} + \lambda i_{6,3*}\{h_{5,9}\} & (k = 5). \end{cases} \end{aligned}$$

Thus $$\pi^4_{k,5} = Z_\infty \quad \textit{when } k \geqslant 9,$$

generated by a such that $p_{k+5,1*}a = 2\{h_{k+4,k+4}\}$;

and $$\pi^4_{5,5} = Z_2 + Z_\infty,$$

generated by $\{i_{6,4}h_{5,9}\}$, and a such that $p_{10,1*}a = 2\{h_{9,9}\}$. Note that in either case $\Delta^{-1}_*(0) = 0$, whence p_{k+5*} is trivial.

(c) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \rightarrow,$$

i.e. $$\rightarrow Z_2 \rightarrow Z_2 \rightarrow \pi^5_{k,5} \rightarrow 0,$$

by §§ 8.3 (d) and (e), and since p_{k+5*} is trivial by (b). Also

$$i^{-1}_{k+5*}(0) = t_{k+5,5*}\,\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+5}\{t_{k+5,5}\} &= h^*_{k+3,k+5}(i_{k+3,1*}\,\mathfrak{p}_*\{h_{k+2,k+3}\} + \lambda i_{k+1,3*}\{h_{k,k+3}\}) \\ &= i_{k+3,1*}\,\mathfrak{p}_*\,h^*_{k+3,k+5}\{h_{k+2,k+3}\} + \lambda i_{k+1,3*}\,h^*_{k+3,k+5}\{h_{k,k+3}\} \\ &= i_{k+3,1*}\,\mathfrak{p}_*\,12\{h_{k+2,k+5}\}, \end{aligned}$$

since $\pi_{k+5}(S^k) = 0$ ($k > 6$), and $i_{6,3*}\,\pi_{10}(S^5) = 0$ ($k = 5$) by § 8.3 (e),

$$= 0, \quad \text{since } \pi^5_{k,4} \text{ is of order two.}$$

Thus $i^{-1}_{k+5*}(0) = 0$ and $$\pi^5_{k,5} = Z_2,$$

generated by $\{i_{k+3,2}\,\mathfrak{p}h_{k+2,k+5}\}$. Note that Δ_* is trivial, whence p_{k+6*} is onto.

9.4. $k = 1$.

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{5*}} \pi_5(S^5) \xrightarrow{\Delta_*} \pi^3_{1,4} \xrightarrow{i_{4*}} \pi^3_{1,5} \to \pi_4(S^5),$$

i.e.

$$\to Z_\infty \to Z_2 \to \pi^3_{1,5} \to 0,$$

by § 8.5 (b). Again $i_{4*}^{-1}(0) \neq 0$ since otherwise the fibring $V_{6,5}/V_{5,4} \to S^5$ would admit a cross-section by Theorem 1.2, which would imply a four-field on S^5, which is impossible by Theorem 1.1 in (**3**). Hence

$$\pi^3_{1,5} = 0 \quad \textit{and} \quad \pi^2_{2,4} = 0,$$

the latter by virtue of Theorem 4.2 (a). But, by Corollary 1.5, $\{t_{6,5}\}$ generates $i_{4*}^{-1}(0)$, $= \pi^3_{1,4}$. Thus from § 8.5 (b) we have that

$$\{t_{6,5}\} = \{i_{4,1}\, \mathrm{p} h_{3,4}\}.$$

Note that the image of Δ_* is of order two, whence the image of p_{5*} is the Z_∞ subgroup generated by $2\{h_{5,5}\}$.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{6*}} \pi_6(S^5) \xrightarrow{\Delta_*} \pi^4_{1,4} \xrightarrow{i_{5*}} \pi^4_{1,5} \xrightarrow{p_{5*}} \pi_5(S^5) \to,$$

i.e.

$$\to Z_2 \to Z_2 \to \pi^4_{1,5} \to Z_\infty \to 0,$$

by § 8.5 (c), and since the image of p_{5*} is a Z_∞ subgroup by (a). Also

$$i_{5*}^{-1}(0) = t_{6,5*}\, \pi_5(S^4),$$

which is generated by

$$h^*_{5,4}\{t_{6,5}\} = h^*_{4,5}\, i_{4,1*}\, \mathrm{p}_*\{h_{3,4}\} = i_{4,1*}\, \mathrm{p}_*\{h_{3,5}\}, \text{ the generator of } \pi^4_{1,4}.$$

Thus $$i_{5*}^{-1}(0) = \pi^4_{1,4},$$

whence $$\pi^4_{1,5} = Z_\infty,$$

generated by $p_{6,1*}^{-1}\, 2\{h_{5,5}\}$. Theorem 4.2 ($a$) then gives that

$$\pi^3_{2,4} = Z_\infty,$$

generated by $p_{6,1*}^{-1}\, 2\{h_{5,5}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{6*} is trivial.

(c) When $p = 5$, (D) gives

$$\xrightarrow{p_{7*}} \pi_7(S^5) \xrightarrow{\Delta_*} \pi^5_{1,4} \xrightarrow{i_{6*}} \pi^5_{1,5} \xrightarrow{p_{6*}} \pi_6(S^5) \to,$$

i.e.

$$\to Z_2 \to 0 \to \pi^5_{1,5} \to 0,$$

by § 8.5 (d), and since p_{6*} is trivial by (b). Thus

$$\pi^5_{1,5} = 0 \quad \textit{and} \quad \pi^4_{2,4} = 0,$$

the latter by virtue of Theorem 4.2 (a). Note that, since Δ_* is trivial, p_{7*} is onto.

(*d*) When $p = 6$ (D) gives

$$\xrightarrow{p_{8*}} \pi_8(S^5) \xrightarrow{\Delta_*} \pi^6_{1,4} \xrightarrow{i_{7*}} \pi^6_{1,5} \xrightarrow{p_{7*}} \pi_7(S^5) \rightarrow,$$

i.e.
$$\rightarrow Z_{24} \rightarrow Z_\infty \rightarrow \pi^6_{1,5} \rightarrow Z_2 \rightarrow 0,$$

by § 8.5 (*e*), and since p_{7*} is onto $\pi_7(S^5)$ by (*c*). Further, $i_{7*}^{-1}(0) = 0$, since it is impossible to map a finite group essentially into an infinite cyclic one. Thus $\pi^6_{1,5}$ *is an extension of* Z_∞ *by* Z_2, *as, by Theorem* 4.2 (*a*), *is* $\pi^5_{2,4}$. Note that Δ_* is trivial, whence p_{8*} is onto.

To calculate the extension we operate with $h^*_{r,r+1}$ on the section of the sequence associated with the fibring $V_{6,4}/S^2 \rightarrow V_{6,3}$ for which $r = 5$ and 6, to obtain the diagram

$$\begin{array}{ccccccccc}
\rightarrow \pi_7(S^2) & \xrightarrow{i_{7*}} & \pi^5_{2,4} & \xrightarrow{p_{7*}} & \pi^4_{3,3} & \xrightarrow{\Delta_{7*}} & \pi_6(S^2) \rightarrow \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* & & \uparrow h^* \\
\rightarrow \pi_6(S^2) & \xrightarrow{i_{6*}} & \pi^4_{2,4} & \xrightarrow{p_{6*}} & \pi^3_{3,3} & \xrightarrow{\Delta_{6*}} & \pi_5(S^2) \rightarrow,
\end{array}$$

which is commutative by Lemma 3.1 (*b*). Using the results of §§ 7.2 (*d*), (*f*), and (*c*) above, the diagram becomes

$$\begin{array}{ccccccccc}
\rightarrow Z_2 & \rightarrow & \pi^5_{2,4} & \rightarrow & Z_\infty + Z_4 & \rightarrow & Z_{12} \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\rightarrow Z_{12} & \rightarrow & 0 & \rightarrow & Z_2 & \rightarrow & Z_2 \rightarrow.
\end{array}$$

Now let a be the generator of $\pi^3_{3,3}$ and a' a generator of order four in $\pi^4_{3,3}$. Then from the exactness of the lower line it follows that

$$\Delta_* a = \{h_{2,5}\}.$$

Hence
$$\Delta_* h^* a = h^* \Delta_* a = h^*_{5,6}\{h_{2,5}\} = 6\{h_{2,6}\}.$$

Thus $h^* a \neq 0$, whence it follows that

$$h^* a = 2a',$$

the only element of order two in $\pi^4_{3,3}$, and that

$$\Delta_* 2a' = 6\{h_{2,6}\} \neq 0.$$

Again,
$$i_{7*}\pi_7(S^2) = i_{7*}h^*\pi_6(S^2) = h^* i_{6*}\pi_6(S^2) = 0,$$

whence p_{7*} is an isomorphism into. Hence, if $\pi^5_{2,4}$ were $Z_\infty + Z_2$, $2a'$ would be in $p_{7*}\pi^5_{2,4}$ since $2a'$ is the only element of order two in $\pi^4_{3,3}$ and so must be the image of the element of order two in $\pi^5_{2,4}$. Thus $\Delta_* 2a'$ would have to be zero, and we have just proved the contrary. Hence

$$\pi^5_{2,4} = Z_\infty,$$

generated by a such that $p_{6,1*}a = \{h_{5,7}\}$, and

$$\pi^6_{1,5} = Z_\infty,$$

generated by a such that $p_{6,1*}a = \{h_{5,7}\}$.

(*e*) When $p = 7$, (D) gives

$$\xrightarrow{p_{9*}} \pi_9(S^5) \xrightarrow{\Delta_*} \pi^7_{1,4} \xrightarrow{i_{8*}} \pi^7_{1,5} \xrightarrow{p_{8*}} \pi_8(S^5) \rightarrow,$$

i.e.
$$\rightarrow Z_2 \rightarrow 0 \rightarrow \pi^7_{1,5} \rightarrow Z_{24} \rightarrow 0,$$

by § 8.5 (*f*), and since p_{8*} is onto $\pi_8(S^5)$ by (*d*). Thus

$$\pi^7_{1,5} = Z_{24},$$

generated by $p^{-1}_{6,1*}\{h_{5,8}\}$. Hence, by Theorem 4.2 (*a*), we have that

$$\pi^6_{2,4} = Z_{24},$$

generated by $p^{-1}_{6,1*}\{h_{5,8}\}$. Note that since Δ_* is trivial, p_{9*} is onto.

9.5. $k \equiv 2 \pmod 4$ *and* $\geqslant 6$.

We first calculate $\{t_{k+5,5}\} \in \pi^3_{k,4}$. We have from § 2.3 (*b*) that

$$p_{k+4,1*}\{t_{k+5,5}\} = 2\{h_{k+3,k+3}\}.$$

Using the result of § 8.2 (*b*) we thus have that $\{t_{k+5,5}\}$ generates a cyclic infinite summand of $\pi^3_{k,4} = Z_{12} + Z_\infty$.

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+4}),$$

i.e.
$$\rightarrow Z_\infty \rightarrow Z_{12} + Z_\infty \rightarrow \pi^3_{k,5} \rightarrow 0,$$

by § 8.2 (*b*). Also $i^{-1}_{k+3*}(0)$ is generated by $\{t_{k+5,5}\}$, i.e. $i^{-1}_{k+3*}(0)$ is a cyclic infinite summand of $\pi^3_{k,4}$. Hence

$$\pi^3_{k,5} = Z_{12},$$

generated by $\{i_{k+1,4}h_{k,k+3}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{k+4*} is trivial.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \rightarrow,$$

i.e.
$$\rightarrow Z_2 \rightarrow Z_2 \rightarrow \pi^4_{k,5} \rightarrow 0,$$

by § 8.2 (*c*), and since p_{k+4*} is trivial by (*a*). Further,

$$i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3}),$$

which is generated by $h^*_{k+3,k+4}\{t_{k+5,5}\}$. To determine this we consider the section of the sequence associated with the fibring $V_{k+4,4}/S^k \rightarrow V_{k+4,3}$ which is of the form

$$\rightarrow \pi_{k+4}(S^k) \rightarrow \pi^4_{k,4} \xrightarrow{p_{k+4,3*}} \pi^3_{k+1,3} \rightarrow.$$

We have from § 2.3 (*b*) that

$$p_{k+4,3*}\{t_{k+5,5}\} = \{t_{k+5,4}\}.$$

Thus
$$p_{k+4,3*}h^*_{k+3,k+4}\{t_{k+5,5}\} = h^*_{k+3,k+4}p_{k+4,3*}\{t_{k+5,5}\}$$
$$= h^*_{k+3,k+4}\{t_{k+5,4}\}$$
$$= 0 \quad \text{by } 8.4\,(b).$$
Thus, since $\pi_{k+4}(S^k) = 0$ $(k \geqslant 6)$,
$$h^*_{k+3,k+4}\{t_{k+5,5}\} = 0.$$
Hence $i^{-1}_{k+4*}(0) = 0$, and so
$$\pi^4_{k,5} = Z_2,$$
generated by $\{i_{k+2,3}\, ph_{k+1,k+4}\}$. Note that Δ_* is trivial, whence p_{k+5*} is onto.

(*c*) When $p = 5$, and $k \geqslant 10$, (D) gives
$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \rightarrow,$$
i.e.
$$\rightarrow Z_2 \rightarrow Z_4 \rightarrow \pi^5_{k,5} \rightarrow Z_2 \rightarrow 0,$$
by § 8.2 (*d*), and since p_{k+5*} is onto $\pi_{k+5}(S^{k+4})$ by (*b*). But we have from Theorem 4.2 (*b*) that
$$\pi^5_{k,5} \approx \pi^4_{k+1,4}$$
$$= Z_8$$
by § 8.4 (*d*). Hence $i^{-1}_{k+5*}(0) = 0$, whence Δ_* is trivial and so p_{k+6*} is onto, and
$$\pi^5_{k,5} = Z_8,$$
generated by a such that $p_{k+5,1*}a = \{h_{k+4,k+5}\}$.

(*d*) When $p = 5$, and $k = 6$, (D) gives
$$\xrightarrow{p_{12*}} \pi_{12}(S^{10}) \xrightarrow{\Delta_*} \pi^5_{6,4} \xrightarrow{i_{11*}} \pi^5_{6,5} \xrightarrow{p_{11*}} \pi_{11}(S^{10}) \rightarrow,$$
i.e.
$$\rightarrow Z_2 \rightarrow Z_\infty + Z_4 \rightarrow \pi^5_{6,5} \rightarrow Z_2 \rightarrow 0,$$
by § 8.2 (*e*), and since p_{11*} is onto $\pi_{11}(S^{10})$ by (*b*). Further,
$$i^{-1}_{11*}(0) = t_{11,5*}\pi_{11}(S^9),$$
which is generated by
$$h^*_{9,11}\{t_{11,5}\} = h^*_{10,11}h^*_{9,10}\{t_{11,5}\} = 0$$
by (*b*) above. Thus $i^{-1}_{11*}(0) = 0$, whence
$$\pi^5_{6,5} \text{ is an extension of } Z_\infty + Z_4 \text{ by } Z_2.$$
Note that Δ_* is trivial, whence p_{12*} is onto.

To determine the extension we consider first the section of the sequence associated with the fibring $V_{11,5}/S^6 \rightarrow V_{11,4}$, which is of the form
$$\rightarrow \pi_{11}(S^6) \rightarrow \pi^5_{6,5} \rightarrow \pi^4_{7,4} \rightarrow \pi_{10}(S^6) \rightarrow,$$

which becomes, with the result of § 8.4 (*d*),

$$\to Z_\infty \to \pi^5_{6,5} \to Z_8 \to 0.$$

From this we see, bearing in mind the result of the last paragraph, that $\pi^5_{6,5}$ is also an extension of Z_∞ by Z_8. Thus we have two possibilities for $\pi^5_{6,5}$: $Z_\infty + Z_8$ or $Z_4 + Z_\infty$. But the method used in §§ 7.32 (*f*) and 8.2 (*e*) does not yield a result in this case, and so all we can say is that

$$\pi^5_{6,5} \textit{ is either } Z_\infty + Z_8 \textit{ or } Z_4 + Z_\infty,$$

generated respectively by $\{i_{7,4} h_{6,11}\}$, and a of order eight such that $p_{11,1*} a = \{h_{10,11}\}$; or $i_{10,1*} a$, where a is of order four and $p_{10,1*} a = \{h_{9,11}\}$, and b such that $p_{11,1*} b = \{h_{10,11}\}$.

9.6. $k \equiv 0 \pmod 4$.

Our first task is to calculate $\{t_{k+5,5}\}$ in $\pi^3_{k,4}$. We have that

$$t_{k+5,5} | S^{k+2} = i_{k+3,1} t_{k+4,4}$$

by § 2.3 (*b*), and that
$$\{t_{k+4,4}\} = 0$$
by § 8.1. Thus we can extend $i_{k+3,1} t_{k+4,4}$ over the hemisphere E^{k+3}_+ of S^{k+3}, and, since $t_{k+4,4}$ is a symmetric map (2.3*a*), we can extend it symmetrically over E^{k+3}_-. Denote this extension by

$$g: S^{k+3} \to i_{k+3,1}(V_{k+3,3}) \subset V_{k+4,4}.$$

Now we use construction 'Q^r' of § 6, with $r = k+3$, $X = V_{k+4,4}$, $f_1 = t_{k+5,5}$, and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\} + \{f_2\} = \{t_{k+5,5}\} + \{g\}.$$

Hence

$$p_{k+4,1*} 2\{h\} = p_{k+4,1*}\{t_{k+5,5}\} + p_{k+4,1*}\{g\} = 2\{h_{k+3,k+3}\},$$

by § 2.3 (*b*) and since $\{g\} \in i_{k+3,1*} \pi^3_{k,3}$. Thus

$$p_{k+4,1*}\{h\} = \{h_{k+3,k+3}\},$$

and
$$\{h\} = \{\mathfrak{p} h_{k+3,k+3}\} + i_{k+3,1*} w, \quad \textit{where } w \in \pi^3_{k,3}.$$

Further, if we consider $\{g\}$ for the moment as in $\pi^3_{k,3}$, we see that $p_{k+3,1} g: S^{k+3} \to S^{k+2}$ is a symmetric map such that, in the notation of § 2.3 (*c*),
$$p_{k+3,1} g u^{-1}_{k+2} = p_{k+3,1} t_{k+4,4} u^{-1}_{k+2}: P^{k+2} \to S^{k+2},$$

which is essential by § 2.3 (*c*); whence $p_{k+3,1} g$ is essential by Theorem 6.1. We thus have, using the results of § 7.31 (*c*), that, in $\pi^3_{k,3}$,

$$\{g\} = (i_{k+1,2*} \bar{x} + \bar{z}),$$

where $\bar{x} \in \pi_{k+3}(S^k)$, and $\bar{z}$ generates a Z_4 summand with

$$2\bar{z} \in i_{k+2,1*} \mathfrak{p}_* \pi_{k+3}(S^{k+1}).$$

Thus

$$\{t_{k+5,5}\} = 2\{h\}-\{g\} = 2\{\mathfrak{p}h_{k+3,k+3}\}+2i_{k+3,1*}w-i_{k+1,3*}\bar{x}-i_{k+3,1*}\bar{z}.$$

Hence $$\{t_{k+5,5}\} = i_{k+1,3*}x+i_{k+3,1*}z+2\mathfrak{p}_*\{h_{k+3,k+3}\},$$

where $x \in \pi_{k+3}(S^k)$ and z generates a Z_4 summand of $\pi^3_{k,3}$ with

$$2z = \{i_{k+2,1}\,\mathfrak{p}h_{k+1,k+3}\}.$$

What remains to be done is to determine x (mod 2) in $\pi_{k+3}(S^k)$. We shall see that there is a distinction between the cases $k \equiv 0$ and 4 (mod 8), quite apart from the special case $k = 4$.

9.61. $k \equiv 0$ (mod 8).

First let us pick generators in $\pi^3_{k,3}$, $\pi^3_{k,4}$, $\pi^4_{k-1,4}$, $\pi^4_{k-1,5}$. By § 7.31 (*c*), $\pi^3_{k,3} = Z_{24}+Z_4$. Pick generators

$\bar{a}$, of order twenty-four, as $i_{k+1,2*}\{h_{k,k+3}\}$;
$\bar{b}$, of order four, as the z defined in § 9.6 above.

By §§ 8.1 and 7.31 (*c*), $\pi^3_{k,4} = Z_{24}+Z_4+Z_\infty$. Pick generators

a, of order twenty-four, as $i_{k+3,1*}\bar{a}$;
b, of order four, as $i_{k+3,1*}\bar{b}$;
c, of infinite order, as $\mathfrak{p}_*\{h_{k+3,k+3}\}$.

By § 8.4 (*d*), $\pi^4_{k-1,4} = Z_8$. Let

$\bar{v}$ be any particular generator.

By § 9.2 (*b*), $\pi^4_{k-1,5} = Z_8+Z_\infty$. Pick generators

v, of order eight, as $i_{k+3,1*}\bar{v}$;
w, of infinite order, as $\{t_{k+5,6}\}$.

This latter choice is possible since

$$p_{k+4,1*}\{t_{k+5,6}\} = 2\{h_{k+3,k+3}\}$$

by § 2.3 (*b*).

Now consider the diagram:

$$\begin{tikzcd} \arrow[r] & \pi_{k+3}(S^{k-1}) \arrow[r, "i_{k,3*}"] & \pi^4_{k-1,4} \arrow[r, "p_{k+3,3*}"] \arrow[dr, "p_{k+3,1*}"'] & \pi^3_{k,3} \arrow[r] \arrow[d, "p_{k+3,1*}"] & {,} \\ & & & \pi_{k+3}(S^{k+2}) & \end{tikzcd}$$

where the horizontal sequence is associated with the fibring

$$V_{k+3,4}/S^{k-1} \to V_{k+3,3},$$

and the triangle is commutative since $p_{k+3,1}p_{k+3,3} = p_{k+3,1}$. Further, since $\pi_{k+3}(S^{k-1}) = 0$, $p_{k+3,3*}$ is a monomorphism. Thus

$$p_{k+3,3*}\bar{v} = 3\epsilon\bar{a}+\mu\bar{b},$$

where $\epsilon \equiv 1 \pmod 2$, and μ unknown. But

$$p_{k+3,1*}\bar{v} = \{h_{k+2,k+3}\},$$

by § 8.4 (*d*) and

$$p_{k+3,1*}\bar{a} = 0, \quad p_{k+3,1*}\bar{b} = \{h_{k+2,k+3}\}, \quad p_{k+3,1*}2\bar{b} = 0,$$

by § 7.31 (*c*). Hence we have by commutability that μ is odd, i.e.

$$p_{k+3,3*}\bar{v} = 3\epsilon\bar{a} \mp \bar{b}, \quad \textit{where } \epsilon \equiv 1 \pmod 2.$$

Now consider the commutative diagram

$$\begin{tikzcd}
\to \pi_{k+3}(S^{k-1}) \arrow[r, "i_{k,3*}"] \arrow[d, "i_{k,0*}"] & \pi^4_{k-1,4} \arrow[r, "p_{k+3,3*}"] \arrow[d, "i_{k+3,1*}"] & \pi^3_{k,3} \arrow[r, "\Delta_*"] \arrow[d, "i_{k+3,1*}"] & \pi_{k+2}(S^{k-1}) \to \arrow[d, "i_{k,0*}"] \\
\to \pi_{k+3}(S^{k-1}) \arrow[r, "i_{k,1*}"] & \pi^4_{k-1,5} \arrow[r, "p_{k+4,4*}"] & \pi^3_{k,4} \arrow[r, "\Delta_*"] & \pi_{k+2}(S^{k-1}) \to,
\end{tikzcd}$$

in which the horizontal sequences are associated with the fibrings $V_{k+3,4}/S^{k-1} \to V_{k+3,3}$ and $V_{k+4,5}/S^{k-1} \to V_{k+4,4}$. Substitution for the groups changes the diagram into

$$\begin{tikzcd}
\to 0 \to \arrow[d] & Z_8 \arrow[r] \arrow[d] & Z_{24}+Z_4 \arrow[r] \arrow[d] & Z_{24} \to \arrow[d] \\
\to 0 \to & Z_8+Z_\infty \arrow[r] & Z_{24}+Z_4+Z_\infty \arrow[r] & Z_{24} \to.
\end{tikzcd}$$

Thus

$$\begin{aligned} p_{k+4,4*}v &= p_{k+4,4*}i_{k+3,1*}\bar{v} = i_{k+3,1*}p_{k+3,3*}\bar{v} \\ &= i_{k+3,1*}(3\bar{a} \mp \bar{b}) \\ &= 3a \mp b. \end{aligned}$$

Further, since by § 2.3 (*b*) $p_{k+4,4*}\{t_{k+5,6}\} = \{t_{k+5,5}\}$, we have from § 9.6 that

$$p_{k+4,4*}w = \lambda a + b + 2c,$$

where λ is to be determined.

Now the next stage of the lower sequence is

$$\xrightarrow{\Delta_*} \pi_{k+2}(S^{k-1}) \to \pi^3_{k-1,5} \to \pi^2_{k,4} \xrightarrow{\Delta_*} \pi_{k+1}(S^{k-1}) \to,$$

which becomes, with the results of §§ 9.2 (*a*) and 8.1 and 7.31 (*b*),

$$\to Z_{24} \to Z_2 \to Z_2 + Z_2 \to Z_2 \to,$$

whence we have by exactness that $\Delta_* \pi^3_{k,4} = \pi_{k+2}(S^{k-1})$.

To recapitulate, the position now is this:

$$0 \to Z_8 + Z_\infty \xrightarrow{p_{k+4,4*}} Z_{24} + Z_4 + Z_\infty \to \quad Z_{24} \quad \to 0$$

generated by v, w, $\quad a$, b, c, $\{h_{k-1,k+2}\}$,

where

$$\begin{aligned} p_{k+4,4*}v &= 3\epsilon a \mp b \qquad (\epsilon \equiv 1 \pmod 2)), \\ p_{k+4,4*}w &= \lambda a + b + 2c \quad (\lambda \textit{ to be determined}). \end{aligned}$$

Hence the factor group $\pi^3_{k,4}/p_{k+4,4*}\pi^4_{k-1,5}$ is generated by a, b, c with the relations:

$$24a = 0, \quad 4b = 0, \quad 3\epsilon a \mp b \equiv 0, \quad \lambda a + b + 2c \equiv 0.$$

Hence $$4(3\epsilon a \mp b) \equiv 0;$$

whence $$12a \equiv 0.$$

Further $$2(3\epsilon a \mp b) \equiv 0;$$

whence $$6a + 2b \equiv 0.$$

But $$6(\lambda a + b + 2c) \equiv 0;$$

whence $$6\lambda' a + 2b + 12c \equiv 0,$$

where

$$\lambda' = 1 \text{ if } \lambda \equiv 1 \pmod 2 \quad \text{and} \quad \lambda' = 0 \text{ if } \lambda \equiv 0 \pmod 2).$$

But, if $\lambda' = 1$, we have, since $6a + 2b \equiv 0$, that $12c \equiv 0$, i.e.

$$12a \equiv 12b \equiv 12c \equiv 0,$$

and the factor group cannot possibly be cyclic of order 24. Hence

$$\lambda' = 0, \quad \text{and thus} \quad \lambda \equiv 0 \pmod 2,$$

i.e. $$\lambda = 2\sigma.$$

Thus $$\{t_{k+5,5}\} = b + 2(\sigma a + c).$$

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e. $$\to Z_\infty \to Z_{24} + Z_4 + Z_\infty \to \pi^3_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (c). But i^{-1}_{k+3*} is generated by $\{t_{k+5,5}\}$, that is by $b + 2(\sigma a + c)$ in the notation of the last paragraph. Hence

$$\pi^3_{k,5} = Z_{24} + Z_8,$$

generated by $\{i_{k+1,4}\,h_{k,k+3}\}$ and $\{i_{k+4,1}\,\mathfrak{p}h_{k+3\,k+3}\}$. Note that, since $\Delta_*^{-1}(0) = 0$, p_{k+4*} is trivial.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e. $$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^4_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (*d*), and since p_{k+4*} is trivial by (*a*). Also

$$i_{k+4*}^{-1}(0) = t_{k+5,5*}\pi_{k+4}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h_{k+3,k+4}^*[b+2(\sigma a+c)] &= h_{k+3,k+4}^* b + 2(\sigma a+c)_*\{h_{k+3,k+4}\} \\ &= h_{k+3,k+4}^* b \\ &= h_{k+3,k+4}^* i_{k+3,1*}\bar{b} = i_{k+3,1*} h_{k+3,k+4}^* \bar{b}, \end{aligned}$$

by Lemma 3.1 (*b*). But

$$\begin{aligned} p_{k+3,1*} h_{k+3,k+4}^* \bar{b} = h_{k+3,k+4}^* p_{k+3,1*}\bar{b} &= h_{k+3,k+4}^*\{h_{k+2,k+3}\} \\ &= \{h_{k+2,k+4}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (*d*), that $i_{k+4*}^{-1}(0)$ is a Z_2 subgroup of $\pi_{k,4}^4$, and that

$$\pi_{k,5}^4 = Z_2 + Z_2,$$

generated by $\{i_{k+2,3}\, \mathfrak{p} h_{k+1,k+4}\}$ and $\{i_{k+4,1}\, \mathfrak{p} h_{k+3,k+4}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^5 \xrightarrow{i_{k+5*}} \pi_{k,5}^5 \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \rightarrow,$$

i.e. $$\rightarrow Z_2 \rightarrow Z_2 + Z_2 \rightarrow \pi_{k,5}^5 \rightarrow 0$$

by § 8.1 and § 7.31 (*e*), and since p_{k+5*} is trivial by (*b*). Further

$$i_{k+5*}^{-1}(0) = t_{k+5,5*}\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h_{k+3,k+5}^*[b+2(\sigma a+c)] &= h_{k+3,k+5}^* b + 2(\sigma a+c)_*\{h_{k+3,k+5}\} \\ &= h_{k+3,k+5}^* b \\ &= h_{k+3,k+5}^* i_{k+3,1*}\bar{b} = i_{k+3,1*} h_{k+3,k+5}^* \bar{b}. \end{aligned}$$

But

$$\begin{aligned} p_{k+3,1*} h_{k+3,k+5}^* \bar{b} = h_{k+3,k+5}^* p_{k+3,1*}\bar{b} &= h_{k+3,k+5}^*\{h_{k+2,k+3}\} \\ &= 12\{h_{k+2,k+5}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (*e*), that $i_{k+5*}^{-1}(0)$ is a Z_2 subgroup of $\pi_{k,4}^5$, and that

$$\pi_{k,5}^5 = Z_2,$$

generated by $\{i_{k+4,1}\, \mathfrak{p} h_{k+3,k+5}\}$. Note that again p_{k+6*} is trivial.

9.62. $k \equiv 4 \pmod 8$ *and* $\geqslant 12$.

We have from § 9.1 that $\pi_{k-1,5}^4 \approx \pi_{k-1,4}^4 + Z_\infty$, where the infinite summand is generated by $\mathfrak{p}_*\{h_{k+3,k+3}\}$, and from § 8.4 (*d*) that $\pi_{k-1,4}^4 = Z_8$.

I shall show in § 10.7 that

$$\{t_{k+5,6}\} = v + 2\mathfrak{p}_*\{h_{k+3,k+3}\} \in \pi^4_{k-1,5},$$

where v generates the finite summand. Thus we can pick generators in $\pi^4_{k-1,5}$ as

v, of order eight, as defined above by $\{t_{k+5,6}\}$;
w, of infinite order, as $\mathfrak{p}_*\{h_{k+3,k+3}\}$.

By §§ 8.1 and 7.31 (c), $\pi^3_{k,4} = Z_{24} + Z_4 + Z_\infty$. Pick generators

a, of order twenty-four, as $i_{k+1,3*}\{h_{k,k+3}\}$;
b, of order four, as $i_{k+3,1*}z$, where z is as defined in 9.6;
c, of infinite order, as $\mathfrak{p}_*\{h_{k+3,k+3}\}$.

Consider now the section of the sequence associated with the fibring $V_{k+4,5}/S^{k-1} \to V_{k+4,4}$, which is of the form

$$\to \pi_{k+3}(S^{k-1}) \to \pi^4_{k-1,5} \xrightarrow{p_{k+4,4*}} \pi^3_{k,4} \to,$$

i.e. $\qquad \to 0 \to Z_8 + Z_\infty \to Z_{24} + Z_4 + Z_\infty \to$

generated by $\qquad v, \quad w, \quad a, \quad b, \quad c.$

Thus we have, since v is of order eight,

$$p_{k+4,4*}v = 3\epsilon a + \mu b, \quad \text{where } \epsilon \equiv 1 \pmod 2,$$

and, since w is of infinite order,

$$p_{k+4,4*}w = \alpha a + \beta b + \gamma c, \quad \text{where } \gamma \neq 0.$$

But, by § 2.3 (b),

$$\begin{aligned}\{t_{k+5,5}\} &= p_{k+4,4*}\{t_{k+5,6}\} \\ &= p_{k+4,4*}(v+2w) \\ &= (3\epsilon + 2\alpha)a + (\mu + 2\beta)b + 2\gamma c.\end{aligned}$$

But we have from § 9.6 that

$$\{t_{k+5,5}\} = \lambda a + b + 2c.$$

Hence

$$\lambda \equiv 1 \pmod 2, \quad \text{i.e. } \lambda = 2\sigma + 1, \quad \mu + 2\beta \equiv 1 \pmod 4, \quad \gamma = 1.$$

Thus $\qquad \{t_{k+5,4}\} = (a+b) + 2(\sigma a + c).$

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e. $\qquad \to Z_\infty \to Z_{24} + Z_4 + Z_\infty \to \pi^3_{k,5} \to 0,$

by §§ 8.1 and 7.31 (c). Also $i^{-1}_{k+3*}(0)$ is generated by $\{t_{k+5,5}\}$: that is, in the notation of the last paragraph, by $(a+b)+2(\sigma a+c)$. Hence

$$\pi^3_{k,5} = Z_4 + Z_{48},$$

generated by $i_{k+3,2*}a$, where

$$p_{k+3,1*}a = \{h_{k+2,k+3}\} \quad\text{and}\quad 2a = \{i_{k+2,1}\,\mathfrak{p}h_{k+1,k+3}\},$$

and
$$\{i_{k+4,1}\,\mathfrak{p}h_{k+3,k+3}\}+\sigma\{i_{k+1,4}\,h_{k,k+3}\}.$$

Note that, since $\Delta_*^{-1}(0) = 0$, p_{k+4*} is trivial.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2 \to \pi^4_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (*d*), and since p_{k+4*} is trivial by (*a*). Also

$$i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+4}[(a+b)+2(\sigma a+c)] &= h^*_{k+3,k+4}(a+b)+2(\sigma a+c)_*\{h_{k+3,k+4}\} \\ &= h^*_{k+3,k+4}\,a+h^*_{k+3,k+4}\,b. \end{aligned}$$

But

$$h^*_{k+3,k+4}\,a = h^*_{k+3,k+4}\,i_{k+1,3*}\{h_{k,k+3}\} = i_{k+1,3*}\,h^*_{k+3,k+4}\{h_{k,k+3}\} = 0.$$

Also, if $b = i_{k+3,1*}\bar{b}$, we have that

$$h^*_{k+3,k+4}\,b = h^*_{k+3,k+4}\,i_{k+3,1*}\bar{b} = i_{k+3,1*}\,h^*_{k+3,k+4}\bar{b},$$

by Lemma 3.1 *b*,

$$\begin{aligned} p_{k+3,1*}\,h^*_{k+3,k+4}\bar{b} = h^*_{k+3,k+4}\,p_{k+3,1*}\bar{b} &= h^*_{k+3,k+4}\{h_{k+2,k+3}\} \\ &= \{h_{k+2,k+4}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (*d*), that $i^{-1}_{k+4*}(0)$ is a Z_2 subgroup of $\pi^4_{k,4}$, and that

$$\pi^4_{k,5} = Z_2+Z_2,$$

generated by $\{i_{k+2,3}\,\mathfrak{p}h_{k+1,k+4}\}$ and $\{i_{k+4,1}\,\mathfrak{p}h_{k+3,k+4}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2 \to \pi^5_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (*e*), and since p_{k+5*} is trivial by (*b*). Also

$$i^{-1}_{k+5*}(0) = t_{k+5,5*}\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+5}[(a+b)+2(\sigma a+c)] &= h^*_{k+3,k+5}(a+b)+2(\sigma a+c)_*\{h_{k+3,k+5}\} \\ &= h^*_{k+3,k+5}\,a+h^*_{k+3,k+5}\,b \\ &= h^*_{k+3,k+5}\,b \end{aligned}$$

since
$$h^*_{k+3,k+5}a = h^*_{k+4,k+5}h^*_{k+3,k+4}a = 0,$$
by (b). Further, if $b = i_{k+3,1*}\bar{b}$, we have
$$h^*_{k+3,k+5}b = h^*_{k+3,k+5}i_{k+3,1*}\bar{b} = i_{k+3,1*}h^*_{k+3,k+5}\bar{b},$$
$$p_{k+3,1*}h^*_{k+3,k+5}\bar{b} = h^*_{k+3,k+5}p_{k+3,1*}\bar{b} = h^*_{k+3,k+5}\{h_{k+2,k+3}\}$$
$$= 12\{h_{k+2,k+5}\}.$$
Thus we see, by looking at the results of §§ 8.1 and 7.31 (e), that $i^{-1}_{k+5*}(0)$ is a Z_2 summand of $\pi^5_{k,4}$, and that
$$\pi^5_{k,5} = Z_2,$$
generated by $\{i_{k+4,1}\mathfrak{p}h_{k+3,k+5}\}$. Note that again p_{k+6*} is trivial.

9.63. $k = 4$.

We first consider the diagram

$$\begin{array}{ccccccc}
\to \pi^4_{3,4} & \xrightarrow{i_{7,1*}} & \pi^4_{3,5} & \xrightarrow{p_{8,1*}} & \pi_7(S^7) \to \\
\downarrow p_{7,3*} & & \downarrow p_{8,4*} & & \downarrow \bar{p}_{8,1*} \\
\to \pi^3_{4,3} & \xrightarrow{i_{7,1*}} & \pi^3_{4,4} & \xrightarrow{p_{8,1*}} & \pi_7(S^7) \to,
\end{array}$$

where the horizontal sequences are those associated with the fibrings $V_{8,5}/V_{7,4} \to S^7$ and $V_{8,4}/V_{7,3} \to S^7$. By § 2.1 the diagram is commutative and $\bar{p}_{8,1*}$ an isomorphism. By §§ 8.1, 9.1 the $i_{7,1*}$ are monomorphisms and the $p_{8,1*}$ are onto. Hence, with the results of §§ 8.1 and 9.1, together with those of §§ 7.31 (c) and 8.4 (e), the diagram becomes

$$\begin{array}{ccccccccccccccc}
0 \to & \underset{\bar{u}}{Z_4} + \underset{\bar{v}}{Z_\infty} & \to & \underset{u}{Z_4} + \underset{v}{Z_\infty} + \underset{w}{Z_\infty} & \to & \underset{\{h_{7,7}\}}{Z_\infty} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \underset{\bar{a}}{Z_\infty} + \underset{\bar{b}}{Z_{12}} + \underset{\bar{c}}{Z_4} & \to & \underset{a}{Z_\infty} + \underset{b}{Z_{12}} + \underset{c}{Z_4} + \underset{d}{Z_\infty} & \to & \underset{\{h_{7,7}\}}{Z_\infty} & \to 0,
\end{array}$$

the summands being generated by the elements displayed below them. These generators are chosen as follows.

In § 10.7 it will be shown that $\{t_{9,6}\} \in \pi^4_{3,5}$ is of the form
$$\{t_{9,6}\} = (2\rho+1)i_{7,1*}\bar{v} + 2\mathfrak{p}_*\{h_{7,7}\},$$
where $\bar{v}$ generates an infinite summand of $\pi^4_{3,4}$. Thus we can choose generators as follows. In $\pi^4_{3,4} = Z_4 + Z_\infty$, choose

$\bar{u}$, of order four (any such);

$\bar{v}$, of infinite order, as defined above by $\{t_{9,6}\}$.

In $\pi^4_{3,5} = Z_4 + Z_\infty + Z_\infty$, choose

u, of order four, as $i_{7,1*}\bar{u}$;
v, of infinite order, as $i_{7,1*}\bar{v}$;
w, of infinite order, as $\mathfrak{p}_*\{h_{7,7}\} + \rho v$.

In $\pi^3_{4,3} = Z_\infty + Z_{12} + Z_4$, choose

$\bar{a}$, of infinite order, as $i_{5,2*}\bar{p}_*\{h_{7,7}\}$ ($\bar{p}$ is Hopf map);
$\bar{b}$, of order twelve, as $i_{5,2*}\mathfrak{E}\{h_{3,6}\}$;
$\bar{c}$, of order four, as the z defined in § 9.6.

In $\pi^3_{4,4} = Z_\infty + Z_{12} + Z_4 + Z_\infty$, choose

a, of infinite order, as $i_{7,1*}\bar{a}$;
b, of order twelve, as $i_{7,1*}\bar{b}$;
c, of order four, as $i_{7,1*}\bar{c}$;
d, of infinite order, as $\mathfrak{p}_*\{h_{7,7}\}$.

Now consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which is of the form

$$\to \pi^4_{3,4} \xrightarrow{p_{7,3*}} \pi^3_{4,3} \xrightarrow{\Delta_*} \pi_6(S^3) \to \pi^3_{3,4} \to \pi^2_{4,3} \to \pi_5(S^3) \to.$$

From the second paragraph of § 8.4 (e) we have that $p_{7,3*}$ is an isomorphism into. Further, since $\pi^3_{3,4} = Z_2$ by § 8.4 (c), $\pi^2_{4,3} = Z_2 + Z_2$ by § 7.31 (b), and $\pi_5(S^3) = Z_2$, it follows by exactness that Δ_* is onto $\pi_6(S^3)$. Now both $\pi^4_{3,4}$ and $\pi^3_{4,3}$ project onto $\pi_7(S^6)$, so that we have the diagram

$$\begin{tikzcd}
\arrow[r] & \pi^4_{3,4} \arrow[rr, "p_{7,3*}"] \arrow[dr, "p_{7,1*}"'] & & \pi^3_{4,3} \arrow[r, "\Delta_*"] \arrow[dl, "p_{7,1*}"] & \pi_6(S^3) \arrow[r] & 0, \\
& & \pi_7(S^6) & & &
\end{tikzcd}$$

which is commutative. Now from § 7.31 (c) we have that

$$p_{7,1*}\bar{b} = 0, \quad p_{7,1*}\bar{c} \neq 0, \quad p_{7,1*}2\bar{c} = 0.$$

Thus, since, by § 8.4 (e), $p_{7,1*}\bar{u} = 0$ and since $\bar{u}$ is of order four,

$$p_{7,3*}\bar{u} = 3\epsilon\bar{b} + 2\mu\bar{c}, \quad \textit{where} \quad \epsilon = \mp 1 \quad \textit{and} \quad \mu = 0 \textit{ or } 1.$$

Again, by § 7.31 (c), $p_{7,1*}\bar{a} = 0$. Thus, since $p_{7,1*}\bar{v} \neq 0$ and $\bar{v}$ is of infinite order,

$$p_{7,3*}\bar{v} = \alpha\bar{a} + \beta\bar{b} + (2\gamma+1)\bar{c}, \quad \textit{where } \alpha \neq 0 \textit{ and } \gamma = 0 \textit{ or } 1.$$

Now change the basis of $\pi^3_{4,3}$ to $\{\bar{a}, \bar{b}_1, \bar{c}_1\}$, where

$$\bar{b}_1 = \bar{b} + 2\mu\bar{c}, \qquad \bar{c}_1 = [2(\gamma - \beta\mu) + 1]\bar{c}.$$

Then the position becomes

$$0 \to Z_4 + Z_\infty \xrightarrow{p_{7,3*}} Z_\infty + Z_{12} + Z_4 \to Z_{12} \to 0$$

generated by $\quad \bar{u} \quad \bar{v} \qquad \bar{a} \quad \bar{b}_1 \quad \bar{c}_1$

with $\quad p_{7,3*}\bar{u} = 3\epsilon\bar{b}_1, \quad where\ \epsilon = \mp 1,$

$$p_{7,3*}\bar{v} = \alpha\bar{a} + \beta\bar{b}_1 + \bar{c}_1, \quad where \quad \alpha \neq 0.$$

Thus the factor group, $\pi^3_{1,3}/p_{7,3*}\pi^4_{3,4}$, is generated by $\bar{a}$, $\bar{b}_1$, $\bar{c}_1$ with the relations

$$12\bar{b}_1 = 0, \quad 4\bar{c}_1 = 0, \quad 3\bar{b}_1 \equiv 0, \quad \alpha\bar{a} + \beta\bar{b}_1 + \bar{c}_1 \equiv 0.$$

There are now two cases

(*a*) $\beta \equiv 0 \pmod 3$, $\qquad$ (*b*) $\beta \equiv \epsilon_1 \pmod 3$, $\quad$ *where* $\epsilon_1 = \mp 1$.

In the case (*a*) we see that the factor group becomes $Z_3 + Z_{4\alpha}$. But the factor group is in fact $Z_{12} = Z_3 + Z_4$. Hence

$$\alpha = 1.$$

In case (*b*), we first change the basis to $\{\bar{a}, \bar{b}_2, \bar{c}_1\}$, where

$$\bar{b}_2 = \epsilon_1 \bar{b}_1 + \bar{c}_1,$$

and then to $\{\bar{a}, \bar{b}_2, \bar{c}_2\}$, where

$$\bar{c}_2 = \bar{c}_1 + 3\bar{b}_2.$$

The factor group will then be generated by $\bar{a}$, $\bar{b}_2$, $\bar{c}_2$, with the relations

$$12\bar{b}_2 = 0, \quad 4\bar{c}_2 = 0, \quad \bar{c}_2 \equiv 0, \quad \alpha\bar{a} + \bar{b}_2 \equiv 0.$$

Thus in this case the factor group becomes $Z_{12\alpha}$. But it is Z_{12}. So again

$$\alpha = 1.$$

Thus we have that

$$p_{7,3*}\bar{v} = \bar{a} + \beta\bar{b} + (2\gamma+1)\bar{c}.$$

Hence

$$p_{8,4*}v = p_{8,4*}i_{7,1*}\bar{v} = i_{7,1*}p_{7,3*}\bar{v}$$
$$= a + \beta b + (2\gamma+1)c.$$

Also, since $p_{8,1*}w = \{h_{7,7}\}$, $p_{8,1*}p_{8,4*}w = \{h_{7,7}\}$. Hence

$$p_{8,4*}w = \alpha' a + \beta' b + \gamma' c + d.$$

But, by § 2.3 (*b*),

$$\{t_{9,5}\} = p_{8,4*}\{t_{9,6}\}$$
$$= p_{8,4*}[(2\rho+1)v + 2\mathfrak{p}_*\{h_{7,7}\}]$$
$$= p_{8,4*}(v+2w).$$

Thus $\quad \{t_{9,5}\} = (2\alpha'+1)a + (2\beta'+\beta)b + [2(\gamma'+\gamma)+1]c + 2d.$

But we have from § 9.6 that

$$\{t_{9,5}\} = \lambda a + \mu b + c + 2d,$$

whence, by comparing coefficients, we see that

$$\lambda = 2\alpha'+1, \qquad 2(\gamma'+\gamma) \equiv 0 \pmod 4,$$

$$\{t_{9,5}\} = a+\mu b+c+2(\sigma a+d) \quad (\sigma = \alpha').$$

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{8*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^3_{4,4} \xrightarrow{i_{7*}} \pi^3_{4,5} \to \pi_7(S^8),$$

i.e.
$$\to Z_\infty \to Z_\infty + Z_{12} + Z_4 + Z_\infty \to \pi^3_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*c*). But $i_{7*}^{-1}(0)$ is generated by $\{t_{9,5}\}$: that is, in the notation of the previous paragraph, by $[(a+\mu b+c)+2(\sigma a+d)]$. Thus

$$\pi^3_{4,5} = Z_{12}+Z_4+Z_\infty,$$

generated by $\{i_{5,4}\mathfrak{E}h_{3,6}\}$, $i_{7,3*}a$, where $p_{7,1*}a = \{h_{6,7}\}$ and $2a = \{i_{6,1}\mathfrak{p}h_{5,7}\}$, and $\{i_{8,1}\mathfrak{p}h_{7,7}\}+\sigma\{i_{5,4}\bar{p}h_{7,7}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{8*} is trivial.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^4_{4,4} \xrightarrow{i_{8*}} \pi^4_{4,5} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2+Z_2+Z_2 \to \pi^4_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*d*), and since p_{8*} is trivial by (*a*). Further

$$i_{8*}^{-1}(0) = t_{9,5*}\,\pi_8(S^7),$$

which is generated by

$$h^*_{7,8}[(a+\mu b+c)+2(\sigma a+d)] = h^*_{7,8}a+\mu h^*_{7,8}b+h^*_{7,8}c+2(\sigma a+d)_*\{h_{7,8}\}$$
$$= h^*_{7,8}a+h^*_{7,8}b+h^*_{7,8}c.$$

But

$$h^*_{7,8}a = h^*_{7,8}i_{5,3*}\bar{p}_*\{h_{7,7}\} = i_{5,3*}\bar{p}_*h^*_{7,8}\{h_{7,7}\} = i_{5,3*}\bar{p}_*\{h_{7,8}\},$$

$$h^*_{7,8}b = h^*_{7,8}i_{5,3*}\mathfrak{E}\{h_{3,6}\} = i_{5,3*}\mathfrak{E}h^*_{6,7}\{h_{3,6}\} = i_{5,3*}\mathfrak{E}\{h_{3,7}\}.$$

Further
$$h^*_{7,8}c = h^*_{7,8}i_{7,1*}\bar{c} = i_{7,1*}h^*_{7,8}\bar{c},$$

$$p_{7,1*}h^*_{7,8}\bar{c} = h^*_{7,8}p_{7,1*}\bar{c} = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Thus, using the results of §§ 8.1 and 7.31 (*d*), we see that $i_{8*}^{-1}(0) \neq 0$, and, since it must be cyclic, that it is a Z_2 summand, and that

$$\pi^4_{4,5} = Z_2+Z_2+Z_2+Z_2,$$

generated by $\{i_{8,1}\mathfrak{p}h_{7,8}\}$, $\{i_{6,3}\mathfrak{p}h_{5,8}\}$, $\{i_{5,4}\mathfrak{E}h_{3,7}\}$, and $\{i_{5,4}\bar{p}h_{7,8}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{9*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{10*}} \pi_{10}(S^8) \xrightarrow{\Delta_*} \pi^5_{4,4} \xrightarrow{i_{9*}} \pi^5_{4,5} \xrightarrow{p_{9*}} \pi_9(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2+Z_2+Z_2 \to \pi^5_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*e*), and since p_{9*} is trivial by (*b*). Further,

$$i_{9*}^{-1}(0) = t_{9,5*}\,\pi_9(S^7),$$

which is generated by

$$h^*_{7,9}[(a+\mu b+c)+2(\sigma a+d)] = h^*_{7,9}a+\mu h^*_{7,9}b+h^*_{7,9}c+2(\sigma a+d)_*\{h_{7,9}\}$$
$$= h^*_{7,9}a+h^*_{7,9}b+h^*_{7,9}c.$$

But

$$h^*_{7,9}a = h^*_{7,9}i_{5,3*}\bar{p}_*\{h_{7,7}\} = i_{5,3*}\bar{p}_*h^*_{7,9}\{h_{7,7}\} = i_{5,3*}\bar{p}_*\{h_{7,9}\},$$
$$h^*_{7,9}b = h^*_{7,9}i_{5,3*}\mathfrak{E}\{h_{3,6}\} = i_{5,3*}\mathfrak{E}h^*_{6,8}\{h_{3,6}\} = i_{5,3*}\mathfrak{E}\{h_{3,8}\}.$$

Further $$h^*_{7,9}c = h^*_{7,9}i_{7,1*}\bar{c} = i_{7,1*}h^*_{7,9}\bar{c},$$

$$p_{7,1*}h^*_{7,9}\bar{c} = h^*_{7,9}p_{7,1*}\bar{c} = h^*_{7,9}\{h_{6,7}\} = 12\{h_{6,9}\}.$$

Thus, by looking at the results of §§ 8.1 and 7.31 (*e*), we see that $i_{9*}^{-1}(0) \neq 0$, whence it must be a Z_2 summand since it is cyclic, and further that

$$\pi^5_{4,5} = Z_2+Z_2+Z_2+Z_2,$$

generated by $\{i_{6,3}\,\mathfrak{p}h_{5,9}\}$, $\{i_{8,1}\,\mathfrak{p}h_{7,9}\}$, $\{i_{5,4}\,\bar{p}h_{7,9}\}$, and $\{i_{5,4}\,\mathfrak{E}_{3,8}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{10*} is trivial.

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SQUARES CONTAINED IN PLANE CONVEX SETS

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THERE are a number of results known which assert that plane convex sets of given minimal width contain sets of certain type and size, and there are similar results for plane convex sets of given constant width [see, for example, (**1**), (**2**), (**3**), (**4**)]. I establish here two results when the contained set is a square:

(*a*) *the minimum width of a convex set that contains a given square is largest when the convex set is a regular triangle*;

(*b*) *the width of a set of constant width that contains a given square is largest when the set of constant width is a Reuleaux triangle.*

If s denotes the side-length of the square and Δ the minimal width in (*a*) and the width in (*b*), then in (*a*) $s \geqslant 2(2-\surd 3)\Delta$ and in (*b*) $s \geqslant 2k$, where k is the positive root of

$$4k^2+(1+2\surd 3)k\Delta+\Delta^2-(4k+\surd 3\Delta)(\Delta^2-k^2)^{\frac{1}{2}} = 0. \qquad (1)$$

Approximately k is $0{\cdot}3237\Delta$.

(*a*) I show first that, if X is an equilateral triangle and s is the side length of a square contained in X, then

$$s \leqslant 2(2-\surd 3)\Delta,$$

where Δ is the minimal width of X, and this inequality is the best possible. By the Blaschke selection-theorem there is a largest square contained in X; denote one such by S. Because of the size of the angles at the vertices it is not possible for a vertex of S to coincide with a vertex of X. Since S is a largest square contained in X, there must be at least one vertex of S on each side of X, Thus either

(i) one side of X contains two vertices of S,

or (ii) there is exactly one vertex of S on each side of X and one vertex of S is interior to X.

Now case (ii) cannot occur, for, if it did, let p be a point such that of the three feet of perpendiculars from p to the sides of X two at least coincide with vertices of S; since the angle made by these perpendiculars at p is 120°, $p \in S$ and thus $p \in X$. Consider a small rotation of S about p. If the sense of this rotation is correctly chosen, S is

transformed into an equal-sized square all of whose vertices are interior points of X. But this contradicts the fact that S is one of the largest squares contained in X. Thus (ii) cannot occur. But then direct calculation leads to the inequality stated above.

Next let X be any closed bounded plane convex set of minimal width Δ. There exists a square S whose vertices belong to fr X [see (**5**), (**6**)]. Let these vertices in order in the clockwise sense round fr X be a, b, c, d and let $\tau_a, \tau_b, \tau_c, \tau_d$ be support lines to X at a, b, c, d. Let the side-length of S be s. Our aim now is to show that

$$s \geqslant 2(2-\sqrt{3})\Delta,$$

and to do this we shall keep S fixed and consider different sets in place of X.

Let l_a, l_b, l_c, l_d be four lines through a, b, c, d respectively which do not cut S. They are then support lines of a convex set

$$W(l_a, l_b, l_c, l_d) = W,$$

which contains S. Denote the minimal width of W by $\Delta(W)$ and the class of all W by w.

Since W contains a circle of radius $\frac{1}{3}\Delta(W)$ and this circle contains a square S_1 oriented in the same manner as S and of side-length $\frac{1}{3}\sqrt{2}\,\Delta(W)$, it follows that

$$\tfrac{1}{3}\sqrt{2}\,\Delta(W) \leqslant s,$$

for otherwise at least one vertex of S would be an interior point of the convex cover of $S \cup S_1$ and hence of W. Thus the numbers $\Delta(W)$ are bounded above. Write

$$\Delta^* = \sup \Delta(W) \quad (W \in w).$$

By a suitable application of the Blaschke selection-theorem there exists a member W^* of w such that

$$\Delta^* = \Delta(W^*).$$

By the argument above about the equilateral triangle we know that

$$\Delta^* \geqslant s/2(2-\sqrt{3}), \tag{2}$$

and thus no two of the lines bounding W^* are parallel unless they also coincide, for otherwise we should have

$$\Delta^* \leqslant \sqrt{2}s,$$

and this contradicts inequality (2). Thus W^* is either a triangle or a quadrilateral and in the latter case no two sides are parallel.

Suppose in the first place that W^* is a quadrilateral and that its vertices are w_1, w_2, w_3, w_4, where the notation is such that a lies on

$w_4 w_1$; b lies on $w_1 w_2$; c lies on $w_2 w_3$; and d lies on $w_3 w_1$ (see Fig. 1). Since no two sides of W^* are parallel, we may assume, with no real loss of generality, that $w_1 w_4$ meets $w_2 w_3$ in u; $w_1 w_2$ meets $w_3 w_4$ in v such that w_1 lies between w_4 and u; w_2 lies between w_3 and u; w_2 lies between w_1 and v; w_3 lies between w_4 and v. We assume that no side of W^* contains more than one of the points a, b, c, d, for, if this were not the case, we could omit one of the lines bounding W^* to obtain an extremal triangle belonging to w. Such triangles are considered later.

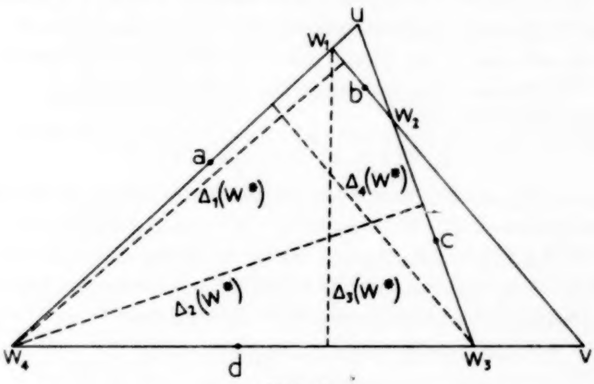

FIG. 1

A width of a polygon can be equal to its minimal width only if it is between two support lines of which one at least meets the polygon in a side. Combining this with the intersections of sides given above we see that the only widths of W^* that can possibly be equal to $\Delta(W^*)$ are the lengths of the following perpendiculars: that from w_4 to $w_1 w_2$; that from w_4 to $w_2 w_3$; that from w_1 to $w_3 w_4$; and that from w_3 to $w_1 w_4$. Denote these lengths by $\Delta_1(W^*)$, $\Delta_2(W^*)$, $\Delta_3(W^*)$, $\Delta_4(W^*)$ respectively. We know that

$$\Delta_i(W^*) \geqslant \Delta(W^*) = \Delta^* \quad (i = 1, 2, 3, 4)$$

and that equality holds for at least one value of i. The next step is to show that equality holds for each $i = 1, 2, 3, 4$.

If $\Delta_3(W^*) > \Delta^*$, let w_3' be a point on the line through w_3 parallel to $w_1 w_4$ such that

$$\tfrac{1}{2}\pi > \angle dcw_3' > \angle dcw_3$$

and the perpendicular distance from w_1 to $w_3' d$ is greater than Δ^*. Suppose further that w_3' is sufficiently close to w_3 for the lines $w_3' c$, $w_3' d$ to meet $w_1 w_2$, $w_1 w_4$ in w_2', w_4' respectively such that $w_1 w_2' w_3' w_4'$ is a

quadrilateral of the class w. Let the perpendicular distance from d to cw_3 be ρ and from d to cw_3' be ρ'. Then, if we denote the quadrilateral $w_1 w_2' w_3' w_4'$ by W_1^*, we have

$$\frac{\Delta_2(W^*)}{\rho} = \frac{w_3 w_4}{w_3 d} = \frac{w_3' w_4'}{w_3' d} = \frac{\Delta_2(W_1^*)}{\rho'}$$

and, since $\rho = dc \sin \angle dcw_3 < dc \sin \angle dcw_3' = \rho'$,

we have $\Delta_2(W_1^*) > \Delta_2(W^*) \geqslant \Delta^*.$

Also $\Delta_1(W_1^*) > \Delta_1(W^*) \geqslant \Delta^*$. Finally, denote by v' the point of intersection of $w_1 w_2'$ with $w_4' w_3'$. Replace w_3' by w_3'', a point between w_3' and v' on $w_4' w_3'$, and denote by w_2'' the point of intersection of $w_3'' c$ and $w_1 b$. Then, if W_2^* denotes the quadrilateral $w_1 w_2'' w_3'' w_4'$, we have, if w_3'' is sufficiently close to w_3',

$$\Delta_i(W_2^*) > \Delta^* \ (i = 1, 2, 3, 4),$$

and hence $\Delta(W_2^*) > \Delta^*$. But $W_2^* \in w$, and thus we have a contradiction to the definition of Δ^*. Thus $\Delta_3(W^*) \leqslant \Delta^*$, i.e. $\Delta_3(W^*) = \Delta^*$.

Next, if $\Delta_1(W^*) > \Delta^*$, replace w_1 by w_1' lying between w_1 and u. Let $w_1' b$ meet $w_3 c$ in w_2' and let W_3^* be the quadrilateral $w_1' w_2' w_3 w_4$. We suppose that w_1' is close to w_1; then $W_3^* \in w$, and

$$\Delta_1(W_3^*) > \Delta^*, \qquad \Delta_3(W_3^*) > \Delta^*.$$

But also $\Delta_2(W_3^*) = \Delta_2(W^*), \qquad \Delta_4(W_3^*) = \Delta_4(W^*).$

Thus $\Delta(W_3^*) \geqslant \Delta(W^*)$. Hence $\Delta(W_3^*) = \Delta^*$, and W_3^* is an extremal quadrilateral of the class w just in the same way that W^* is. But then the argument above applied to W_3^* leads to a contradiction as before. Thus the assumption is false and $\Delta_1(W^*) = \Delta^*$. Similarly

$$\Delta_2(W^*) = \Delta_4(W^*) = \Delta^*.$$

Since $\Delta_1(W^*) = \Delta_3(W^*)$ and $\Delta_2(W^*) = \Delta_4(W^*)$, we have

$$\angle w_2 w_1 w_4 = \angle w_1 w_4 w_3, \qquad \angle w_1 w_4 w_3 = \angle w_2 w_3 w_4.$$

Since $\Delta_1(W^*) = \Delta_2(W^*)$, we have $\angle w_1 w_2 w_4 = \angle w_3 w_2 w_4$. Thus in the triangles $w_1 w_2 w_4$ and $w_3 w_2 w_4$ we have

$$\angle w_2 w_1 w_4 = \angle w_2 w_3 w_4, \qquad \angle w_1 w_2 w_4 = \angle w_3 w_2 w_4,$$

and the side $w_2 w_4$ is common; it follows that these triangles are congruent and the quadrilateral W^* is symmetrical in the line $w_2 w_4$. In these circumstances it is possible to construct a square $a'b'c'd'$ whose vertices belong to the sides of W^* with a' on $w_1 w_4$, b' on $w_1 w_2$, c' on $w_2 w_3$, d' on $w_3 w_4$ such that $a'b'$ is parallel to $w_4 w_2$ and $a'b'c'd'$ is symmetric about $w_2 w_4$. Denote the side-length of this square by s'.

We show next that $s' \leqslant s$. This is a consequence of the following lemma:

LEMMA. *If l_1 and l_2 are two fixed lines intersecting in the point X, and if Y_1 and Y_2 are variable points on l_1, l_2 respectively such that the distance $Y_1 Y_2$ is fixed, then the area of the triangle XY_1Y_2 is a concave function of the angle $\angle Y_1 Y_2 X$ for*

$$\tfrac{1}{2}(\tfrac{1}{2}\pi - \angle Y_1 X Y_2) < \angle Y_1 Y_2 X < \tfrac{1}{2}(\tfrac{3}{2}\pi - \angle Y_1 X Y_2).$$

Let O be the circumcentre of the triangle XY_1Y_2 and R its circumradius. As Y_1, Y_2 vary, R, and therefore $\angle OY_1Y_2$ also, are fixed. The area of the triangle is equal to

$$\tfrac{1}{2} Y_1 Y_2 \{R \sin(2\angle Y_1 Y_2 X - \angle OY_1 Y_2) + R \sin \angle OY_1 Y_2\}$$

and is therefore a concave function of the angle $\angle Y_1 Y_2 X$ in the range given.

It follows also from the above formula that the area of the triangle XY_1Y_2 is a maximum when $XY_1 = XY_2$.

Now let a_2', b_1' be points on $w_4 w_1$, $w_1 w_2$ respectively such that $a_2' b_1'$ is parallel to $w_2 w_4$ and the length of the segment $a_2' b_1'$ is s. Similarly construct three other segments of length s, say $b_2' c_1'$, $c_2' d_1'$, $d_2' a_1'$, where $b_2' c_1'$, $d_2' a_1'$ are perpendicular to $w_2 w_4$ and $c_2' d_1'$ is parallel to $w_2 w_4$, b_1', b_2' lie on $w_1 w_2$; c_1', c_2' on $w_2 w_3$; d_1', d_2' on $w_3 w_4$; a_1', a_2' on $w_1 w_4$. If $s' > s$, the segments $a_2' b_1'$, $b_2' c_1'$, $c_2' d_1'$, $d_2' a_1'$ divide W^* into four triangles and an eight-sided figure Y. The area of the eight-sided figure Y is greater than that of the square $a'b'c'd'$ and this in turn is greater than the area of the square $abcd$. By the lemma, observing that the three relevant angles are in the range of the lemma, we have

$$2 \operatorname{area} w_1 a_2' b_1' \geqslant \operatorname{area} w_1 ab + \operatorname{area} w_3 cd.$$

By the remark after the lemma,

$$\operatorname{area} w_4 d_2' a_1' \geqslant \operatorname{area} w_4 da,$$

$$\operatorname{area} w_2 b_2' c_1' \geqslant \operatorname{area} w_2 bc.$$

Adding these inequalities we get that

$$\begin{aligned} &\operatorname{area} Y + \operatorname{area} w_1 a_2' b_1' + \operatorname{area} w_3 c_2' d_1' + \operatorname{area} w_4 d_2' a_1' + \operatorname{area} w_2 b_2' c_1' \\ &\qquad > \operatorname{area} abcd + \operatorname{area} w_1 ab + \operatorname{area} w_2 bc + \operatorname{area} w_3 cd + \operatorname{area} w_4 da. \end{aligned}$$

But this is impossible since each side of the inequality is equal to the area of W^*. Hence $s' \leqslant s$.

Direct calculation shows that

$$\Delta(W^*) = s'(\sin 3\alpha + \cos \alpha), \quad \text{where } \alpha = \angle w_1 w_4 w_2.$$

Since $\angle w_1 w_2 w_4 = \pi - 3\alpha \leqslant \frac{1}{2}\pi$, we have $\alpha \geqslant \frac{1}{6}\pi$; also, since $w_1 w_4$ meets $w_2 w_3$ in the manner prescribed, $\alpha < \frac{1}{4}\pi$. But as α increases from $\frac{1}{6}\pi$ to $\frac{1}{4}\pi$, $\Delta(W^*)/s'$ decreases.

Hence $$\Delta(W^*) \leqslant \tfrac{1}{2}s'(2+\sqrt{3}).$$

Thus $$\Delta^* \leqslant \tfrac{1}{2}s(2+\sqrt{3}),$$

and this is the required inequality.

Suppose in the second place that W^* is a triangle with vertices w_1, w_2, w_3 such that a lies on $w_3 w_1$, b on $w_1 w_2$, and both c and d on $w_2 w_3$. Symmetrize the triangle about the perpendicular bisector of cd. We obtain a new triangle W_1^* with vertices w_1', w_2', w_3' such that a lies on $w_3' w_1'$, b on $w_1' w_2'$, and c and d on $w_2' w_3'$. Then, since the area of W_1^* equals that of W^* and the greatest side-length of W_1^* is less than or equal to the greatest side-length of W^*, we have

$$\Delta(W_1^*) \geqslant \Delta(W^*).$$

Now W_1^* is an isosceles triangle with $w_1' w_2' = w_1' w_3'$. A simple calculation shows that the minimal width of W_1^* is less than or equal to that of an equilateral triangle W_2^* with vertices $w_1'' w_2'' w_3''$, where a lies on $w_3'' w_1''$, b on $w_1'' w_2''$, and c and d on $w_2'' w_3''$. Then we have

$$\tfrac{1}{2}s(2+\sqrt{3}) = \Delta(W_1^*) \geqslant \Delta(W_2^*) \geqslant \Delta(W^*) = \Delta^*,$$

and the required inequality is proved.

Since $\Delta(X) \leqslant \Delta^*$, we have established the result (a). For (b) we show first that the largest square of side-length s contained in a Reuleaux triangle of width Δ is such that $s = 2k$, where k is given by (1). Let a, b, c be the vertices of the Reuleaux triangle R and let w_1, w_2, w_3, w_4 be the vertices of a largest square S contained in R. The four vertices w_1, w_2, w_3, w_4 must all lie on the frontier of R, for otherwise, if w_4, say, is an interior point of R and w_1 lies on ab, w_2 on bc, then let p be the point of intersection of cw_1 and aw_2. Give the square S a small rotation about p and each of its vertices will describe an arc of a circle of radius less than Δ. Thus, whatever the sense of rotation, w_1 and w_2 will move into interior points of R and so will w_3 if the sense of rotation is chosen appropriately. This means that there is a square of the same size as S all of whose vertices are interior points of R. Since this is in contradiction to the fact that S is a largest square contained in R, we conclude that all the vertices of S belong to the frontier of R.

Suppose that w_1 lies on ab, w_2 and w_3 on bc, and w_4 on ca. Let t be the mid-point of $w_2 w_3$. Then R is symmetric about at, for at is perpendicular to $w_2 w_3$ and therefore bisects $w_1 w_4$, and this implies that

arc ac is the reflection of arc ab in at. Calculation then shows that S has the side-length stated.

Next let X be a set of constant width Δ and S be a square whose vertices belong to the frontier of X. It will be shown that, if S has side-length s', where $s' < 2k$ and k is given by (1), then X cannot be of constant width Δ.

The following lemma is required:

LEMMA. *Let Δ be a given positive number and x a variable number satisfying*

$$(\surd 3-1)\Delta \leqslant 2\surd 2x \leqslant \Delta.$$

Let a, b be two points in the plane distant $2x$ apart and let m be the mid-point of the segment ab and f be a point on the perpendicular bisector of ab distant $\surd(\Delta^2-x^2)-2x$ from m. Then

(i) *there is a point p on the side of ab opposite to f distant Δ from both f and a,*

(ii) *$\angle mfp$ decreases as x increases.*

(i) Let p be a point distant Δ from f and a on the same side of af as m.

The distance of f from a is less than Δ, and triangle afp is isosceles; hence $\angle fap > 60°$. But the distance $\surd(\Delta^2-x^2)-2x$ is less than or equal to $\surd 3x$ and thus $\angle fam \leqslant 60°$. Hence p lies on the side of ab opposite to f.

(ii) Calculation shows that as x increases af decreases in length. Thus $\angle fap$ increases. But $\angle fam$ decreases and therefore $\angle map$ increases. Now

$$\Delta \cos \angle map - x = \Delta \sin \angle mfp$$

and thus, as x increases, $\angle mfp$ decreases.

In what follows every circle and every circular disk is of radius Δ. The circle with centre p is denoted by C_p.

Denote the vertices of S by a, b, c, d and let the circular disks bounded by C_a, C_b, C_c, C_d intersect in a closed set K. Since X is of diameter Δ and contains a, b, c, d, we have $X \subset K$. Let the arc of C_a which forms part of fr K be denoted by α and define β, γ, δ with respect to C_b, C_c, C_d respectively. Let α, β meet in h; β, γ in i; γ, δ in f; δ, α in g. Let C_f meet α in g_2 and β in i_1; C_g meet β in h_2 and γ in f_1; C_h meet γ in i_2 and δ in g_1, and finally C_i meet δ in f_2 and α in h_1 (see Fig. 2).

Since X is a complete set, each arc α, β, γ, δ contains at least one point of X. On α no point of X can be interior to the arc $g_2 h_1$ for, if there were such a point q, then the intersection L of all the circular disks that contain q, b, d contains c as an interior point. But $L \subset X$

and c is a frontier point of X; thus no point of X lies interior to the arc $g_2 h_1$. Thus there must be a point of X either on the arc hh_1 or on the arc gg_2. Similarly there must be another point of X on the arc ff_1 or on the arc ii_2.

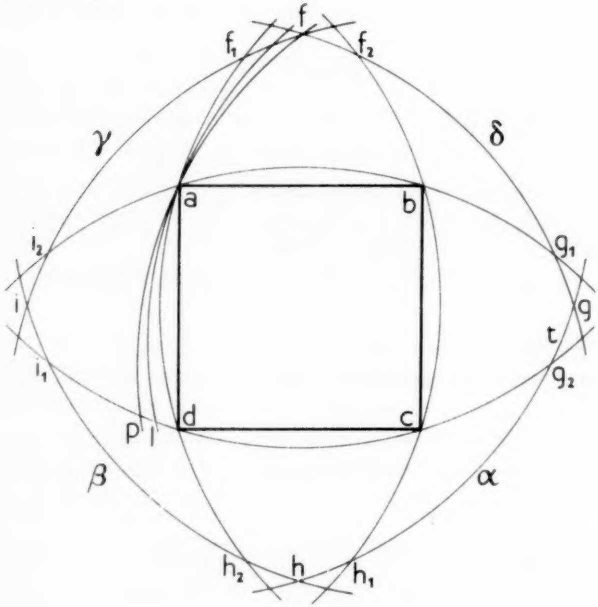

FIG. 2

Suppose that there is a point of X on the arc gg_2, say t. Let C_{g_2} and C_t meet C_f in p and l respectively on the same sides of fg_2 and ft as is d. By the lemma, since $s' < 2k$, the arc length of $g_2 i_1$ is greater than $\frac{1}{3}\pi\Delta$, and thus p lies on this arc. Also l lies between p and d and the arc $i_1 i$ of β is exterior to C_t. Hence no point of X lies on $i_1 i$.

Also C_t meets γ in a point of the arc ff_1, and the arc $i_2 i$ is exterior to C_t. Hence there is a point of X on ff_1. But then an argument similar to the above with the arc ff_1 in place of the arc gg_2 leads to the conclusion that no point of X lies on the arc hh_2. But this is a contradiction, for it implies that no point of X lies on β.

We are led to a similar contradiction if we assume that there is a point of X on the arc hh_1 instead of on gg_2. Thus our original assumption that $s' < 2k$ is false, and so $s' \geqslant 2k$.

Thus (b) is proved.

Remark. The results (*a*) and (*b*) may be rephrased to say that

(*a*) any set of minimal width Δ contains a square of side-length s with $s \geqslant 2(2-\sqrt{3})\Delta$,

(*b*) any set of constant width Δ contains a square of side-length δ with $s \geqslant 2k$, where k is the positive root of (1).

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ON A GENERALIZATION OF BESSEL FUNCTIONS AND A RESULTING CLASS OF FOURIER KERNELS

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1. THIS note contains information about a class of symmetrical Fourier kernels given by A. P. Guinand in this Journal in 1949 [see (**1**)]. These kernels, $G_k(x)$ in the notation used here, are extensions of the Fourier cosine and sine kernels and are given in (**1**) in the form

$$G_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{k} A_r \exp\left(-x\cos\frac{r\pi}{2k}\right)\cos\left(x\sin\frac{r\pi}{2k}+m_r\frac{\pi}{2k}\right) \quad (0 \leqslant x < \infty),$$

where the A_r are constants, the m_r integers, and k is a positive integer.†

Just as the sine and cosine functions satisfy a linear differential equation, of order 2, it is shown here that the kernels $G_k(x)$ satisfy a differential equation of order $2k$, together with certain initial conditions at $x = 0$ (see Theorem 1).

That $(2/\pi)^{\frac{1}{2}}\sin x$ is a Fourier kernel may be considered as a special case, i.e. $\nu = \frac{1}{2}$, of the result that the Bessel function $x^{\frac{1}{2}}J_\nu(x)$ is a Fourier kernel (the Hankel kernel) for $\nu \geqslant -\frac{1}{2}$. A generalization of the Guinand kernels $G_k(x)$ is given here which, in some aspects, is similar to this extension of the sine kernel to the Hankel kernel. The functions resulting from this generalization have properties similar to the Bessel functions; in particular they satisfy linear differential equations of even order, but greater than two. These results are stated in Theorem 2.

Some remarks are made at the end about the connexion between these results and expansions of functions in eigenfunction theory. A possible connexion with the separability of partial differential equations is also raised.

To avoid unnecessary repetition a knowledge of the paper (**1**) by Guinand is assumed; also required are some results from the general theory of symmetrical Fourier kernels given by E. C. Titchmarsh in his book on Fourier Integrals [see (**2**) Chapter VIII].

† One example of a Guinand kernel for $k = 2$ is $\pi^{-\frac{1}{2}}(e^{-x}-\cos x+\sin x)$.

Quart. J. Math. Oxford (2) 10 (1959), 270-9.

2. Let $\qquad G_k(x) \equiv G_k(x; n_1, n_2,..., n_k) \quad (0 \leqslant x < \infty)$

denote the Guinand kernel [see (**1**)] of order k arising from the set of integers $\{n_1, n_2,..., n_k\}$, i.e. a set which satisfies the conditions

$$0 \leqslant n_r \leqslant 2k-1 \quad (1 \leqslant r \leqslant k),$$

$$n_r \neq n_s, \quad n_r \neq 2k-1-n_s \quad (1 \leqslant r, s \leqslant k). \tag{2.1}$$

Our results are contained in the following theorems:

THEOREM 1. (i) *$G_k(x)$ is a solution of the differential equation*†

$$y^{(2k)}+(-1)^{k+1}y = 0 \quad (0 \leqslant x < \infty), \tag{2.2}$$

(ii) *for* $0 \leqslant p \leqslant 2k-1$,

$$[G_k^{(p)}(x)]_{x=0} \begin{cases} = 0 & \textit{if } p = n_r \quad (1 \leqslant r \leqslant k), \\ \neq 0 & \textit{if } p \neq n_r. \end{cases} \tag{2.3}$$

THEOREM 2. *Let $g_k(x)$ $(k \geqslant 1)$ be defined by*

$$g_k(x) = \left(\frac{\pi}{2k}\right)^{\frac{1}{2}} G_k(x; 0, 1, 2,..., k-1) \quad (0 \leqslant x < \infty), \tag{2.4}$$

and further define for real or complex ν

$$J_{\nu,k}(x) = \frac{2k}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\left(\frac{x}{2k}\right)^{\nu} \int\limits_0^1 g_k^{(1)}(xt)(1-t^{2k})^{\nu-\frac{1}{2}}\, dt \quad (\mathrm{re}\,\nu > -\tfrac{1}{2}), \tag{2.5}$$

then (i) *if* $Y \equiv x^{-\nu}J_{\nu,k}(x)$,

$$Y^{(2k)}+\frac{2k(\nu+\frac{1}{2})}{x}Y^{(2k-1)}+(-1)^{k+1}Y = 0, \tag{2.6}$$

(ii) *$x^{\frac{1}{2}}J_{\nu,k}(x)$ is a symmetrical Fourier kernel for the following values of ν and k:*

$$(1) \quad k = 1,\ \nu > -\tfrac{1}{2}, \tag{2.7}$$

$$(2) \quad k > 1,\ \nu = 0 \textit{ and } \nu = \tfrac{1}{2} \textit{ only}. \tag{2.8}$$

Note. We consider $J_{\nu,k}(x)$ only when $\mathrm{re}\,\nu > -\frac{1}{2}$ to secure convergence of (2.5); this restriction can be removed by the use of contour integrals but this is not considered in this note. [Compare G. N. Watson (**3**), chapter VI.]

The function $J_{\nu,k}(x)$ is one possible generalization of the Bessel function $J_\nu(x)$. In fact it is readily seen that, since [from (**1**)]

$$G_1(x; 0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin x,$$

† $y^{(r)}$ will denote the rth differential coefficient of y with respect to x including the case $r = 1$.

we have $g_1(x) = \sin x$,† and so $J_{\nu,1}(x)$ is identical with the Bessel function $J_\nu(x)$ [see (**3**) 48]. Thus the information in (2.7) is a restatement of the Hankel theorem [see (**2**) § 8.18] but is inserted for completeness. The differential equation (2.6) reduces to a transformation of Bessel's equation when $k = 1$.

It is soon verified, from (2.5) and use of (2.3) with $p = 0$, that

$$x^{\frac{1}{2}}J_{\frac{1}{2},k}(x) = G_k(x;\, 0, 1, 2,..., k-1) \quad (k \geqslant 1); \tag{2.9}$$

in this sense the function $x^{\frac{1}{2}}J_{\nu,k}(x)$ is a generalization of the Guinand kernels. The information in (2.8), for $\nu = \frac{1}{2}$, is thus a restatement of the result in (**1**); the interest arises in the isolated case $\nu = 0$ when $k > 1$.

A possible reason for the restriction of the values of ν in (2.8) to ensure that $x^{\frac{1}{2}}J_{\nu,k}(x)$ $(k > 1)$ is a symmetrical Fourier kernel is given in the last section.

3. In this section I give the proof of Theorem 1. This follows from the explicit formula for $G_k(x)$ given in [(**1**) 192], i.e.

$$G_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{k} A_r \exp\left(-x\cos\frac{r\pi}{2k}\right)\cos\left(x\sin\frac{r\pi}{2k} + m_r\frac{\pi}{2k}\right), \tag{3.1}$$

in which the A_r are constants and the m_r integers. From this it is readily verified that, for $0 \leqslant p \leqslant 2k$,

$$G_k^{(p)}(x) = (-1)^p\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{k} A_r \exp\left(-x\cos\frac{r\pi}{2k}\right)\cos\left(x\sin\frac{r\pi}{2k} + m_r\frac{\pi}{2k} - \frac{rp\pi}{2k}\right),$$

and hence (3.2)

$$[G_k^{(p)}(x)]_{x=0} = (-1)^p\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{k} A_r \cos\left(m_r\frac{\pi}{2k} - \frac{rp\pi}{2k}\right). \tag{3.3}$$

If $\mathcal{K}_k(s)$ denotes the Mellin transform of $G_k(x)$, then from (2) and (3) of (**1**) we have

$$\mathcal{K}_k(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(s) \sum_{r=0}^{k} A_r \cos\left(m_r\frac{\pi}{2k} + \frac{rs\pi}{2k}\right) \tag{3.4}$$

$$= 2^k(2\pi)^{-\frac{1}{2}}\Gamma(s) \prod_{r=1}^{k} \sin(s+n_r)\frac{\pi}{2k}. \tag{3.5}$$

Comparing these expressions we see that

$$[G_k^{(p)}(x)]_{x=0} = (-1)^p 2^k (2\pi)^{-\frac{1}{2}} \left\{\prod_{r=1}^{k} \sin(s+n_r)\frac{\pi}{2k}\right\}_{s=-p},$$

and (2.3) of Theorem 1 now follows.

† This is the reason for the notation used in (2.4); it seems to give the best extension of the established notation for Bessel functions [see (**3**) Chapter III].

It is clear from (3.2) above that

$$G_k^{(2k)}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{r=0}^{k} (-1)^r A_r \exp\left(-x\cos\frac{r\pi}{2k}\right)\cos\left(x\sin\frac{r\pi}{2k}+m_r\frac{\pi}{2k}\right), \quad (3.6)$$

so that (2.2) of Theorem 1 would follow from this if the following result were true:

$A_r = 0$ when r is even and k is odd, or when r is odd and k is even.

This does actually hold; for the A_r are obtained by expanding

$$\prod \sin\left\{(s+n_r)\frac{\pi}{2k}\right\}$$

to give (3.4) from (3.5), and, when k is even, it is soon seen that only the A_r with even coefficients appear and vice versa when k is odd.

This completes the proof of Theorem 1.

4. We come now to Theorem 2 and start by discussing the differential equation

$$Y^{(2k)}+2k(\nu+\tfrac{1}{2})x^{-1}Y^{(2k-1)}+(-1)^{k+1}Y = 0 \quad (k \geqslant 1). \quad (2.6)$$

It is not difficult to see that, if x is a complex variable, then every point of (2.6) is regular except for a regular singularity at $x = 0$ and an irregular singularity at infinity; this holds for $k \geqslant 1$.

Let $\eta(x)$ be any solution of the generalized Fourier equation

$$\eta^{(2k)}+(-1)^{k+1}\eta = 0 \quad (0 \leqslant x < \infty) \quad (4.1)$$

with the added condition that

$$[\eta^{(2k-1)}(x)]_{x=0} = 0. \quad (4.2)$$

Define $Y(x)$ by

$$Y \equiv \int_0^1 \eta(xt)(1-t^{2k})^{\nu-\frac{1}{2}} \quad (\operatorname{re}\nu > -\tfrac{1}{2}). \quad (4.3)$$

Then Y is a solution of the equation (2.6). To see this we have

$$Y^{(2k-1)} = \int_0^1 \eta^{(2k-1)}(xt)t^{2k-1}(1-t^{2k})^{\nu-\frac{1}{2}}\,dt,$$

and integrating by parts we get, using (4.2),

$$Y^{(2k-1)} = \frac{x}{2k(\nu+\frac{1}{2})}\int_0^1 \eta^{(2k)}(xt)(1-t^{2k})(1-t^{2k})^{\nu-\frac{1}{2}}\,dt,$$

and from this and use of (4.1) it is clear that Y is a solution of (2.6).

Since there are $2k-1$ linearly independent solutions of (4.1) satisfying the condition (4.2), this leads to the formation of $2k-1$ linearly independent solutions of (2.6). A final solution of (2.6) can be obtained

but is complicated in the general case; compare the integral representations of the Bessel function Y_ν in (**3**). In the case when k is even, say $k = 2p$, a final solution is

$$\int_1^\infty e^{-xt}(t^{2k}-1)^{\nu-\frac{1}{2}}\,dt \quad (\operatorname{re}\nu > -\tfrac{1}{2};\ k = 2p).$$

When $k = 1$, the differential equation (2.6) is equivalent to the Bessel equation and can be transformed into the following canonical form by considering $y = x^{\nu+\frac{1}{2}}Y$ [see for example (**4**) 17]:

$$y^{(2)}+\{1-(\nu^2-\tfrac{1}{4})x^{-2}\}y = 0. \tag{4.4}$$

The canonical form of the general equation (2.6) can also be obtained by considering $y = x^{\nu+\frac{1}{2}}Y$; for example, if $k = 2$, we obtain

$$y^{(4)}-6(\nu-\tfrac{1}{2})(\nu+\tfrac{1}{2})x^{-2}y^{(2)}+8(\nu-\tfrac{1}{2})(\nu+\tfrac{1}{2})(\nu+\tfrac{3}{2})x^{-3}y^{(1)}-$$
$$-3(\nu-\tfrac{1}{2})(\nu+\tfrac{1}{2})(\nu+\tfrac{3}{2})(\nu+\tfrac{5}{2})x^{-4}y-y = 0, \tag{4.5}$$

and a similar result can be obtained for general k.

It is possible to show that the canonical form of (2.6) has the following self-adjoint properties [see (**5**) for self-adjoint equations]:

(i) if $k = 1$, it is self-adjoint for all real ν,

(ii) if $k > 1$, it is self-adjoint *only* for $\nu = -\frac{1}{2}, 0, \frac{1}{2}$.

These should be compared with (2.7) and (2.8).

The equation (2.6) is equivalent to the generalized Fourier equation (4.1) when $\nu = \pm\frac{1}{2}$.

The function $J_{\nu,k}(x)$ seems to have properties similar to the Bessel function $J_\nu(x)$, i.e. recurrence relations and asymptotic expansions.

5. To prove (i) of Theorem 2 it only remains to indicate that, from the results of Theorem 1, $g_k^{(1)}(x)$, defined in (2.4), satisfies the equation (4.1) and the initial condition (4.2). Thus from the integral representation (2.5) it follows that $x^{-\nu}J_{\nu,k}(x)$ is a solution of the differential equation (2.6).

To prove (ii) of Theorem 2 we apply the Mellin transform test to $x^{\frac{1}{2}}J_{\nu,k}(x)$ to decide whether or not this function is a symmetrical Fourier kernel [for this test see (**2**) §§ 8.1 and 2]: that is, we calculate the Mellin transform of $x^{\frac{1}{2}}J_{\nu,k}(x)$, say $\mathscr{K}_{\nu,k}(s)$,

$$\mathscr{K}_{\nu,k}(s) = \int_0^\infty x^{s-1}x^{\frac{1}{2}}J_{\nu,k}(x)\,dx \quad (\operatorname{re}\nu > -\tfrac{1}{2};\ k \geqslant 1) \tag{5.1}$$

and then determine whether or not it satisfies the functional equation

$$\mathscr{K}_{\nu,k}(s)\mathscr{K}_{\nu,k}(1-s) = 1. \tag{5.2}$$

We find, on making the calculation, that

$$\mathscr{K}_{\nu,k}(s) = (2k)^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{4k+1}{4k}-\frac{s}{2k}-\frac{\nu}{2k}\right)}{\Gamma\left(\frac{2k+1}{4k}-\frac{s}{2k}+\frac{2k-1}{2k}\nu\right)} \prod_{n=0}^{k-1} \frac{\Gamma\left(\frac{s}{2k}+\frac{\nu}{2k}+\frac{2k-1}{4k}+\frac{n}{2k}\right)}{\Gamma\left(\frac{4k+1}{4k}-\frac{s}{2k}-\frac{\nu}{2k}-\frac{n}{2k}\right)}, \tag{5.3}$$

and that the integral (5.1) is convergent for†

$$\tfrac{1}{2}-\nu-k < \operatorname{re} s < 1. \tag{5.4}$$

The details of this calculation are left to the next section.

We can now complete Theorem 2. When $k = 1$, we have

$$\mathscr{K}_{\nu,1}(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})},$$

which is the Mellin transform of the function $x^{\frac{1}{2}}J_\nu(x)$ [see (**2**) § 8.4]; it is clear that (5.2) is satisfied by this function, certainly when $\nu > -\frac{1}{2}$.

When $k > 1$, it can be similarly verified from (5.3) that the functional equation (5.2) is satisfied when $\nu = 0$ and $\nu = \frac{1}{2}$; in the latter case we return to the Guinand kernel G_k. For other values of ν, (5.2) is not satisfied.‡

This completes the discussion of Theorem 2.

6. The calculation of $\mathscr{K}_{\nu,k}(s)$ is tedious, and I outline the steps required omitting some of the details. We start formally by inverting the repeated integral for $\mathscr{K}_{\nu,k}$, i.e.

$$C \int_0^\infty x^{s-1}\,dx \int_0^1 x^{\nu+\frac{1}{2}} g_k^{(1)}(xt)(1-t^{2k})^{\nu-\frac{1}{2}}\,dt, \tag{6.1}$$

and using the results

(i) if $\mathscr{K}(s)$ is the Mellin transform of $f(x)$, then $(1-s)\mathscr{K}(s-1)$ is the transform of $f^{(1)}(x)$;

(ii) from (2.4) above and equation (2) of (**1**), the Mellin transform of $g_k(x)$ is

$$\left(\frac{\pi}{2k}\right)^{\frac{1}{2}} 2^k(2\pi)^{-k}\Gamma(s) \prod_{n=0}^{k-1} \sin(s+n)\frac{\pi}{2k};$$

(iii) the Gauss multiplication theorem for $\Gamma(2kz)$, with

$$z = k^{-1}(\tfrac{1}{2}s+\tfrac{1}{2}\nu-\tfrac{1}{4}),$$

† The form of (5.3) and the region of convergence (5.4) imply that the theorems of Chapter VIII of (**2**) are valid for the functions considered in this note.

‡ In particular it is not satisfied when $\nu = -\frac{1}{2}$ $(k > 1)$.

[see (**6**) 225] and the functional equation

$$\sin \pi z = \pi\{\Gamma(z)\Gamma(1-z)\}^{-1}.$$

Use of these gives the result (5.3) above for $\mathscr{K}_{\nu,k}(s)$. To verify this the following lemma is required:

LEMMA.† *There exist constants* $K_r(\nu,k)$ $(r = 1,2,3)$ *dependent only on* ν *and* k, *such that for* $\nu > -\frac{1}{2}$ *and* $k \geqslant 1$,

(i) $|x^{\frac{1}{2}}J_{\nu,k}(x)| < K_1 x^{\nu+k-\frac{1}{2}} \quad (0 \leqslant x \leqslant 1),$ (6.2)

(ii) *for* $0 \leqslant p \leqslant 2k$,

$$|\{x^{\frac{1}{2}}J_{\nu,k}(x)\}^{(p)}| < K_2 \quad (0 \leqslant x < \infty), \tag{6.3}$$

(iii) $\left|\int\limits_0^X x^{\frac{1}{2}}J_{\nu,k}(x)\,dx\right| < K_3 \quad (0 \leqslant X < \infty).$ (6.4)

The proof of (6.2) follows from considering $x^{-\nu}J_{\nu,k}(x)$ as a regular function of x at the origin and using the initial conditions (2.3).

I outline the proof of (6.3) for the special case‡ $k = 1$, $p = 0$, and $-\frac{1}{2} < \nu < \frac{1}{2}$§ and then indicate the extension for other values of k, p, ν. It is clearly sufficient to prove the result for, say, $x \geqslant 1$. We have

$$x^{-\nu}J_{\nu,1}(x) = C\int\limits_0^1 \cos xt(1-t^2)^{\nu-\frac{1}{2}}\,dt$$

and, writing

$$\int\limits_0^1 \cos xt(1-t^2)^{\nu-\frac{1}{2}}\,dt = \int\limits_0^{1-\delta} + \int\limits_{1-\delta}^1 = I_1 + I_2 \quad \text{(say)},$$

where $\delta = x^{-1}$, we get

$$I_1 = \left[\frac{\sin xt}{x}(1-t^2)^{\nu-\frac{1}{2}}\right]_0^{1-\delta} + \frac{2(\nu-\frac{1}{2})}{x}\int\limits_0^{1-\delta} t\sin xt(1-t^2)^{\nu-\frac{3}{2}}\,dt$$

and thus

$$|I_1| \leqslant 2^{\nu-\frac{1}{2}}\delta^{\nu-\frac{1}{2}}x^{-1} + 2(\nu-\tfrac{1}{2})2^{\nu-\frac{3}{2}}x^{-1}\int\limits_0^{1-\delta}(1-t)^{\nu-\frac{3}{2}}\,dt = 2^{\nu-\frac{1}{2}}x^{-1}.$$

Also
$$|I_2| \leqslant 2^{\nu-\frac{1}{2}}\int\limits_{1-\delta}^1 (1-t)^{\nu-\frac{1}{2}}\,dt = \frac{2^{\nu-\frac{1}{2}}}{(\nu+\frac{1}{2})x^{\nu+\frac{1}{2}}}.$$

† These results show how closely the properties of $J_{\nu,k}(x)$, with $k \geqslant 2$, follow those of $J_\nu(x)$.

‡ In the case $k = 1$ the results follow at once from the asymptotic properties of the Bessel functions, but such expansions are not available for higher values of k. They could presumably be obtained by considering contour integrals.

§ The results follow at once when $\nu = \frac{1}{2}$.

From these inequalities (6.3) now follows, for this case, since

$$-\tfrac{1}{2} < \nu < \tfrac{1}{2}.$$

We can extend the range of ν on further integration by parts in the integral I_1. The next integration extends the results up to $\nu < \frac{3}{2}$.

If $k = 1$ and $p > 0$, the only change is a positive power of t in the integrand, and the same method suffices.

When $k > 1$, the term $\cos xt$ is replaced by a sum of similar terms some of which even have small exponential factors [see (3.1)]. The term $(1-t^2)^{\nu-\frac{1}{2}}$ changes to

$$(1-t^{2k})^{\nu-\frac{1}{2}} = (1-t)^{\nu-\frac{1}{2}}\{p(t)\}^{\nu-\frac{1}{2}},$$

where $p(t)$ is a polynomial which does not vanish in $[0, 1]$. The argument now proceeds as above.

The proof of (6.4) follows from (6.3) and the canonical form of the differential equation given in § 4. For example, if $k = 1$ and we write $y = x^{\frac{1}{2}}J_{\nu,1}(x)$, then from (4.4)

$$y = y^{(2)} + \frac{\nu^2-\frac{1}{4}}{x^2-(\nu^2-\frac{1}{4})}y^{(2)}$$

for, say, $x \geqslant X_0 > |\nu^2-\frac{1}{4}|^{\frac{1}{2}}$. Integrating this over the interval $[X_0, X]$ and using (6.3) as required we now obtain (6.4).

There is a similar proof of (6.4) in the general case $k > 1$, using the general canonical form of which (4.5) is an example (for $k = 2$).

7. The validity of the steps used to calculate $\mathscr{K}_{\nu,k}(s)$ now follows from the lemma of the previous section.

From the inequalities (6.2 and 6.3) it follows that the integral

$$\int_0^\infty x^{s-1}x^{\frac{1}{2}}J_{\nu,k}(x)\,dx \tag{7.1}$$

is absolutely convergent for $\frac{1}{2}-\nu-k < \operatorname{re} s < 0$. The convergence can be extended (non-absolutely) to $\operatorname{re} s < 1$ by using (6.4) and the Dirichlet test for convergence of integrals. There is uniform convergence for any closed region inside the strip $\frac{1}{2}-\nu-k < \operatorname{re} s < 1$.

The inversion of the integral (6.1) in the case $k = 1$ need not be considered since the value of $\mathscr{K}_{\nu,1}(s)$ is known [see (**3**) § 13.24].

When $k > 1$, write the integral (6.1) in the form

$$\int_0^1 (...)\,dx \int_0^1 (...)\,dt + \int_1^\infty (...)\,dx \int_0^1 (...)\,dt.$$

The first of these repeated integrals is absolutely convergent for

$\frac{1}{2}-\nu-k < \operatorname{re} s < 1$ and so can be inverted, by the Fubini theorem, in this strip.

The second repeated integral is certainly absolutely convergent for $\operatorname{re}(s+\nu-\frac{1}{2}) < -1$ (since, by (3.2), $g_k^{(1)}(xt)$ is uniformly bounded for all x and t) and so can be inverted for this range.

The common region of inversion is the strip

$$\tfrac{1}{2}-\nu-k < \operatorname{re} s < -\nu-\tfrac{1}{2},$$

and the width of this is $k-1$, which is strictly positive since $k > 1$. In this strip the expression (5.3) for $\mathcal{K}_{\nu,k}(s)$ is valid and this can now be extended to the region

$$\tfrac{1}{2}-\nu-k < \operatorname{re} s < 1$$

by analytical continuation.

8. It is clear that to regard $x^{\frac{1}{2}}J_{\nu,k}(x)$ as a symmetrical Fourier kernel is a generalization of the Hankel theorem for the Bessel function $x^{\frac{1}{2}}J_\nu(x)$ [see (**2**) § 8.18]. In his book on eigenfunction theory E. C. Titchmarsh has shown that the Hankel formula can also be obtained as an eigenfunction expansion [see (**4**) § 4.11]. The same result seems to hold for the generalized function $x^{\frac{1}{2}}J_{\nu,k}(x)$ ($k > 1$; $\nu = 0, \frac{1}{2}$); I have shown that this is certainly the case when $k = 2$, but the details are not given here.

It seems likely that this is the explanation of the fact that, when $k > 1$, the values for which $x^{\frac{1}{2}}J_{\nu,k}(x)$ is a symmetrical kernel are $\nu = 0$ and $\nu = \frac{1}{2}$ since these are the *only* values which satisfy the condition $\operatorname{re}\nu > -\frac{1}{2}$ and for which the attendant differential equation is self-adjoint. The chance is remote of obtaining a symmetrical eigenfunction expansion when the differential equation is not self-adjoint.

The value $\nu = -\frac{1}{2}$ also gives a self-adjoint equation, but, even under extended definition to this value, $x^{\frac{1}{2}}J_{\nu,k}(x)$ is not a symmetrical kernel (see the footnote in § 5). On the other hand the differential equation does possess solutions which are symmetrical kernels when $\nu = -\frac{1}{2}$, i.e. all the Guinand kernels since then the equation reduces to the generalized Fourier equation.

The function $x^{\frac{1}{2}}J_{\nu,k}(x)$ is not the only generalization of the Guinand kernel G_k. In the case $k = 2$ other forms have again been discussed by A. P. Guinand in (**7**) [see in particular 447, 448]. This generalization depends on the Bessel functions J_ν, Y_ν, K_ν, and the resulting kernels again have interesting connexions with fourth-order differential equations.

As far as the author can determine, there are no other symmetrical

Fourier kernels associated with the differential equation (2.6). It seems more than likely that there are pairs of unsymmetrical kernels similar to the pair in [(**2**) 215 example (2)].

Finally the connexion between Bessel functions and the solution of Laplace's equation in polar-coordinate form is well known. One might ask if there is any connexion between the generalized function $J_{\nu,k}(x)$ and partial differential equations of order $2k$.

In the case $k = 2$ two equations which might be of interest are the biharmonic equations (in two or three variables)

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4} = 0 \tag{8.1}$$

and the equation

$$\frac{\partial^4\phi}{\partial x^4}+\frac{\partial^4\phi}{\partial y^4} = 0. \tag{8.2}$$

It would be of interest to know in what coordinate-systems these two equations are separable; as they stand, (8.2) is separable, but (8.1) is not. Are there other coordinate-systems in which they are separable and, if so, does the differential equation (2.6) with $k = 2$, or a transformation of it, play any part?

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A FUNCTION-THEORETIC SOLUTION OF CERTAIN INTEGRAL EQUATIONS (II)

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1. Introduction

In the first part of this paper (**1**) a method of solution was found for the integral equation

$$\int\limits_0^\infty K(|x-\xi|)f(\xi)\,d\xi = g(x) \quad (x > 0), \tag{1.1}$$

with g given and f sought, when $K(w)$ is an analytic function with an isolated, logarithmic singularity at $w = 0$. More precisely it was assumed that

$$K(w) = P(w)\log w + Q(w), \tag{1.2}$$

where P and Q were entire functions of w^2. The methods employed differed from the usual Wiener–Hopf theory in that the Fourier integral theorem was not used. Here and in what follows it is important to distinguish between what we call *Fourier* and *Laplace* transforms. By the *Fourier* transform of $m(x)$ we mean the integral

$$\int\limits_{-\infty}^{+\infty} e^{-iwx}m(x)\,dx \quad (x \text{ real, } w \text{ complex}),$$

provided that the integral has a meaning, while by the *Laplace* transform we mean, for an analytic function $m(z)$,

$$\int\limits_{0\,P}^{\infty} e^{-z\xi}m(z)\,dz$$

along some path P for which the integral converges.

The purpose of this second part is to indicate two ways in which the methods of (**1**) can be extended. The first concerns the equation

$$f(x)+\int\limits_0^\infty K(x-\xi)f(\xi)\,d\xi = g(x) \quad (x > 0), \tag{1.3}$$

where K has the same form as in (1.2). For equations of the second kind it becomes necessary to study the Laplace transforms of multiple-valued functions. We illustrate (1.3) with the special equation

$$f(x)+\pi^{-1}\int\limits_0^\infty K_0(k|x-\xi|)f(\xi)\,d\xi = 0 \quad (x > 0;\ 0 < k < 1), \tag{1.4}$$

where K_0 is the singular Bessel function of second kind. The equation arises in the study of diffraction of water waves obliquely incident on a semi-infinite dock (**2**). The underlying boundary problem is the determination of a function satisfying the differential equation

$$u_{xx}+u_{yy}-k^2u = 0 \quad (y < 0)$$

together with the boundary conditions

$$u_y(x,0) = 0 \quad (x < 0), \qquad u_y(x,0) = u(x,0) \quad (x > 0).$$

Further $u(x,y)$ must possess for large positive x a certain 'wave-like' behaviour. The boundary problem was solved in (**3**) without recourse to the integral equation (1.4), and physical details can be found there.

Equation (1.3) and the equations considered in (**1**) can be solved for certain cases by the methods of Wiener and Hopf. We turn next to a type which cannot be so solved. Since our method does not use Fourier-integral concepts, the solution procedure is freed of the use of the convolution theorem and thus becomes applicable to sum kernels. The analogue of (1.1), that is

$$\int_0^\infty K(x+\xi)f(\xi)\,d\xi = g(x) \quad (x > 0),$$

where K has the form (1.2), presents some new complications in that a solution does not always exist, and it is our intention to discuss these questions at some future time. The analogue of (1.3) is

$$f(x)+\int_0^\infty K(x+\xi)f(\xi)\,d\xi = g(x) \quad (x > 0). \tag{1.5}$$

With K again taken as in (1.2), we can treat (1.5) by our methods but again we need the use of the Laplace transform. To present its solution here would duplicate much of the work connected with (1.3) while at the same time confusing the problems peculiar to sum kernels. Accordingly we have chosen to present equation (1.5) with a kernel of the form

$$K(w) = w^{-1}P(w)+Q(w), \tag{1.6}$$

P and Q again being entire functions. This situation illustrates clearly the differences in method between difference kernels and sum kernels without extraneous complications. To simplify further the calculations we deal with the homogeneous equation (1.5) and more particularly with the equation discussed in (**4**), namely

$$f(x) = \int_0^\infty (x+\xi)^{-1}\exp\{-(x+\xi)\}f(\xi)\,d\xi \quad (x > 0), \tag{1.7}$$

which was treated in (**4**) by an iteration process. The simplest equation of the form we consider, that is

$$f(x) = \int_0^\infty (x+\xi)^{-1} f(\xi)\, d\xi,$$

was solved in (**5**), and our procedure for (1.7) is an extension of that of (**5**).

The methods we employ, like those of (**5**), belong to function theory and are variations of the work of Carleman (**6**) and (**7**). The essence of the method is the translation of the integral equation into a relation describing the nature of the singularities of an analytic function of a complex variable. This relation, together with information about the growth properties for large values of the variable, permits the explicit determination of this analytic function and, from this, the determination of the solution of the integral equation. In this connexion we remark that one of the principal difficulties arising in equations (1.4) and (1.7) as well as in those of (**1**) is that we must admit the possibility of exponential growth.

Although we consider only homogeneous integral equations in this paper, it will become apparent what modifications would be required to treat non-homogeneous equations. A careful reading will reveal the appropriate procedure for equation (1.5) with a logarithmic kernel of the form (1.2).

2. Difference kernels

In this section we consider the equation (1.4),

$$f(x)+\pi^{-1} \int_0^\infty K_0(k|x-\zeta|) f(\zeta)\, d\zeta = 0 \quad (x > 0). \tag{2.1}$$

Here $K_0(w)$ is the singular Bessel function of second kind: that is

$$K_0(w) = -I_0(w)\log w + Q(w), \tag{2.2}$$

where $I_0(w)$ and $Q(w)$ are entire functions of w^2, and $\log w$ is that branch which is real for $\arg w = 0$.

Suppose $f(x)$ a solution of (2.1). Then, for $z \neq 0$, $-2\pi < \arg z < 0$, define an analytic function of $z = x+iy$ by the integral

$$F(z) = \int_0^\infty K_0\{k(\zeta-z)\} f(\zeta)\, d\zeta. \tag{2.3}$$

$F(z)$ is continuous in $0 \leqslant |z|$, $-2\pi \leqslant \arg z \leqslant 0$ when defined in terms of its limit values on $\arg z = -2\pi$ and 0. However, these last are not

equal, for, by (2.2),

$$F(z) = \int_0^\infty K_0\{k|\zeta-z|\}f(\zeta)\,d\zeta-\pi i\int_0^z I_0\{k(\zeta-z)\}f(\zeta)\,d\zeta \tag{2.4}$$

on $\arg z = 0$, while

$$F(z) = \int_0^\infty K_0\{k|\zeta-z|\}f(\zeta)\,d\zeta+\pi i\int_0^z I_0\{k(\zeta-z)\}f(\zeta)\,d\zeta \tag{2.5}$$

on $\arg z = -2\pi$. We show now that (2.4) and (2.5) permit the analytic continuation of $F(z)$, as a multiple-valued function, to a Riemann surface with the origin as logarithmic branch point. On this surface let z^+ and z^- denote the image points of z: that is

$$z^+ = z\exp(2\pi i), \qquad z^- = z\exp(-2\pi i).$$

Then from (2.1), (2.4), and (2.5) we can write, for $\arg z = 0$,

$$F(z^-)-F(z) = 2\pi i\int_0^z I_0\{k(\zeta-z)\}f(\zeta)\,d\zeta. \tag{2.6}$$

Thus, on $\arg z = 0$,

$$F(z)-F(z^-)+i\int_0^z I_0\{k(\zeta-z)\}\{F(\zeta)+F(\zeta^-)\}\,d\zeta = 0. \tag{2.7}$$

We can use (2.7) to continue $F(z)$ to the whole of the logarithmic Riemann surface since, for $F(z)$ in $-4\pi \leqslant \arg z \leqslant -2\pi$, (2.7) is a non-homogeneous Volterra equation for $F(z)$ in terms of values of F in $-2\pi \leqslant \arg z \leqslant 0$. Thus $F(z)$ becomes defined on the entire surface with (2.7) holding everywhere.†

It is clear that for the solution of the integro-difference equation (2.7) for $F(z)$ it is sufficient to solve (2.1) since $f(x)$ can be determined by solving the Volterra equation (2.6). We shall solve (2.7) by taking Laplace transforms. The transform of $I_0(kw)$ is $(s^2-k^2)^{-\frac{1}{2}}$; hence, if we let $\mathscr{F}(s)$ and $\mathscr{F}^*(s)$ denote the Laplace transforms of $F(z)$ and $F(z^-)$ respectively, we obtain, by applying the convolution theorem to (2.7),

$$\mathscr{F}^*(s) = \frac{(s^2-k^2)^{\frac{1}{2}}+i}{(s^2-k^2)^{\frac{1}{2}}-i}\mathscr{F}(s). \tag{2.8}$$

It is convenient at this stage to study the asymptotic behaviour of $F(z)$ for large z. We have,

$$K_0(w) \sim Aw^{-\frac{1}{2}}\exp(-w) \quad (|w| \to \infty;\ -\pi \leqslant \arg w \leqslant +\pi), \tag{2.9}$$

† It is to be noted that in the limiting case $k = 0$, which corresponds to direct incidence of the waves on the dock, the integro-difference equation (2.7) goes over into the difference-differential equation formulated in (8) by different methods.

so that, if $\exp(-k\xi)f(\xi)$ is integrable, we have, formally, from (2.3) that

$$F(z) \sim Bz^{-\frac{1}{2}}\exp(kz) \quad (|z| \to \infty;\ -2\pi < \arg z < 0). \tag{2.10}$$

Hence, if we introduce $G(z) = \exp(-kz)F(z)$,

$$G(z) \to 0 \quad (|z| \to \infty;\ -2\pi < \arg z < 0).$$

On multiplying (2.7) by $\exp(-kz)$ we obtain, then,

$$G(z) - G(z^-) + i\exp(-kz)\int\limits_0^z I_0\{k(\zeta-z)\}\{F(\zeta)+F(\zeta^-)\}\,d\zeta = 0. \tag{2.11}$$

We wish to take the Laplace transform of (2.11), and some preliminary remarks are in order. Let $g(s)$ denote the Laplace transform of $G(z)$. Assuming, for the present, that $G(z)$ remains bounded we can write

$$g(s) = \int\limits_0^\infty \exp(-zs)G(z)\,dz \quad (\arg z = 0). \tag{2.12}$$

The function defined by (2.8) is analytic in $-\frac{1}{2}\pi < \arg s < \frac{1}{2}\pi$. Further, since $G(z)$ is analytic on the logarithmic Riemann surface, we can continue $g(s)$ to the same surface by shifting the path of integration. In particular, to continue $g(s)$ into $\arg s > \frac{1}{2}\pi$, we shift the path into $\arg z < 0$ keeping always $\arg zs = 0$: that is, we write

$$g(s) = \int\limits_{0\,P_s}^\infty \exp(-zs)G(z)\,dz \quad (\arg z = -\arg s),$$

where P_s is the ray $\arg z = -\arg s$. Since $G(z)$ has only $z = 0$ as a singularity, this process of rotating the path may be continued indefinitely in both directions. From our construction we have

$$g(s^+) = \int\limits_{P_s} \exp(-zs)G(z^-)\,dz, \tag{2.13}$$

where s^+ has the same meaning as z^+. Thus the Laplace transform of $G(z^-)$ is $g(s^+)$.

We return now to (2.11). Since multiplication by e^{-kz} reflects in translation by $+k$ in the transform plane, application of the convolution theorem to (2.11) yields

$$g(s) - g(s^+) + i(s^2+2ks)^{-\frac{1}{2}}\{\mathscr{F}(s+k) + \mathscr{F}^*(s+k)\} = 0,$$

i.e., by (2.8),

$$g(s) - g(s^+) + i(s^2+2ks)^{-\frac{1}{2}}\left\{1 + \frac{(s^2+2ks)^{\frac{1}{2}}+i}{(s^2+2ks)^{\frac{1}{2}}-i}\right\}\mathscr{F}(s+k) = 0.$$

But $\mathscr{F}(s+k) = g(s)$, since $G(z) = \exp(-kz)F(z)$, and thus we obtain

$$g(s) - g(s^+) + i(s^2+2ks)^{-\frac{1}{2}}\left\{1 + \frac{(s^2+2ks)^{\frac{1}{2}}+i}{(s^2+2ks)^{\frac{1}{2}}-i}\right\}g(s) = 0,$$

i.e.
$$g(s^+) = \frac{(s^2+2ks)^{\frac{1}{2}}+i}{(s^2+2ks)^{\frac{1}{2}}-i}\,g(s). \tag{2.14}$$

Upon taking the logarithms of (2.14) we see that the general solution of this equation is of the form

$$g(s) = g_0(s)\exp\{h(s)\}, \tag{2.15}$$

where $g_0(s)$ is any single-valued function, and $h(s)$ satisfies

$$h(s^+)-h(s) = \log\left\{\frac{(s^2+2ks)^{\frac{1}{2}}+i}{(s^2+2ks)^{\frac{1}{2}}-i}\right\}. \tag{2.16}$$

One sees from (2.14) that, if $g(s)$ is analytic on one sheet of the Riemann surface, it will have poles on the next sheet. Now $F(z)$ and hence also $G(z)$ are free of singularities for $z \neq 0$ in $-2\pi \leqslant \arg z \leqslant 0$. Moreover $G(z) \to 0$ as $|z| \to \infty$ in this sector of the Riemann surface. From our construction it follows that $g(s)$ is without singularities in the corresponding region of the s-plane: that is, $s \neq 0$, $0 < \arg s < 2\pi$. Further the boundedness of the solution of (2.1) at $x = 0$ implies the continuity of $F(z)$ and thus also of $G(z)$ as $|z| \to 0$. But then the Abel theorem for Laplace transforms implies $g(s) \to 0$ as $|s| \to \infty$.

The requirements thus placed on the functions $h(s)$ and $g_0(s)$ are the following: $h(s)$ should satisfy (2.16), have no singularities in $0 < \arg s < 2\pi$, and remain bounded as $|s| \to \infty$ in this sector. On the other hand, $g_0(s)$ can have singularities only at $s = 0$, where, being single-valued, it can have only poles. Further, $g_0(s)$ must vanish as $|s| \to \infty$. The requirements for $h(s)$ are satisfied by choosing

$$h(s) = (2\pi i)^{-1}\int\limits_0^\infty \frac{1}{t-s}\log\frac{(t^2+2tk)^{\frac{1}{2}}+i}{(t^2+2tk)^{\frac{1}{2}}-i}\,dt \tag{2.17}$$

in $0 \leqslant \arg s \leqslant 2\pi$ and continuing $h(s)$ to successive sheets of the Riemann surface by (2.16) [cf. (**1**)]. As to $g_0(s)$ it must have the form

$$g_0(s) = \sum_{n=1}^{\infty} \alpha_n s^{-n}.$$

Accordingly we shall consider the set of solutions $g^n(s)$ of (2.14) defined by

$$g^n(s) = s^{-n}\exp\{h(s)\}.$$

We shall verify in what follows that the special function g^1 is one satisfactory choice of $g_0(s)$ and assuming this fact for the moment we wish to point out that it is the only one. Let $G^n(z)$ denote those functions which have as transforms $g^n(s) = s^{-n}\exp\{h(s)\}$. Assuming $g^1(s)$ satisfactory we see from (2.10) that

$$G^1(z) = \exp(-kz)F(z) \sim Az^{-\frac{1}{2}} \quad (|z| \to \infty;\ -2\pi < \arg z < 0). \tag{2.18}$$

But we have

$$\int_0^\infty \exp(-zs)\left\{\int_0^z G^n(s)\,ds\right\} dz = s^{-1}\int_0^\infty \exp(-zs)G^n(z)\,dz$$
$$= s^{-1}g^n(s) = g^{n+1}(s).$$

Thus, for $n \geqslant 2$, the $g^n(s)$ correspond to iterated integrals of $G^1(z)$ from 0 to z, and by (18) these become infinite as $|z| \to \infty$. Thus $G^1(z)$ is the only function among the G^n which can satisfy (2.18). We now proceed to verify that $g^1(s) = s^{-1}\exp h(s)$ is a suitable choice.

The solution $f(z)$ of (2.1) can be expressed in terms of g^1 by means of equation (2.6). Multiplying (2.6) by $\exp(-kz)$ yields

$$\exp(-kz)\int_0^z I_0\{k(\zeta-z)\}f(\zeta)\,d\zeta = (2\pi i)^{-1}\{G^1(z^-)-G^1(z)\}.$$

Hence, if we write $L(s)$ for the Laplace transform of $f(z)$, we obtain

$$\frac{L(s+k)}{\sqrt{(s^2+ks)}} = (2\pi i)^{-1}\{g^1(s^+)-g^1(s)\},$$

i.e., from (2.14),

$$L(s+k) = \frac{(s^2+2ks)^{\frac{1}{2}}g^1(s)}{\pi\{(s^2+2ks)^{\frac{1}{2}}-i\}} = \frac{(s^2+2ks)^{\frac{1}{2}}g^1(s^+)}{\pi\{(s^2+2ks)^{\frac{1}{2}}+i\}}. \tag{2.19}$$

Since $L(s+k)$ is the Laplace transform of $\exp(-kz)f(z)$, we can recover $f(z)$ by forming the inverse transform of (2.19). For this purpose we need to know the singularities of $L(s+k)$. By (2.17) and (2.16) we see that $h(s)$ and hence $\exp\{h(s)\}$ have no singularities in $0 \leqslant \arg s \leqslant 2\pi$; $s \neq 0$. It follows that the only singularity of $g^1(s^+)$ in $-2\pi < \arg s < 0$ is $s = 0$. From the second equality of (2.19) it follows that the only singularities of $L(s+k)$ in $-2\pi < \arg s < 0$ occur at

$$s = 0, \qquad s = 2k\exp(-\pi i)$$

and the points at which $\sqrt{(s^2+2ks)}+i$ vanishes. These latter points occur at

$$s_1 = \exp(-i\theta), \qquad s_2 = \exp\{i(\theta-2\pi)\},$$

where

$$\cos\theta = -k, \qquad \sin\theta = \sqrt{(1-k^2)},$$

at which $L(s+k)$ has poles of first order. Next we consider the region $0 < \arg s < 2\pi$, using the first equality of (2.19). By our previous remarks the only singularity of $g(s)$ is at $s = 0$. Hence, in $0 < \arg s < 2\pi$, $L(s+k)$ has singularities only at $s = 0$, $s = 2k\exp\pi i$ and points where $\sqrt{(s^2+2ks)}-i$ vanishes. Clearly the latter are the points s_1^+ and s_2^+.

Collecting our information we say that $L(s+k)$ has singularities in the sector $-\pi \leqslant \arg s^- \leqslant +\pi$ of the Riemann surface at $s = 0$,

$s = -2ke^{\frac{1}{2}\pi i}$, s_1, and s_2^+, the last two being poles. For $\arg z = 0$, we can accordingly form the inverse of (2.19), using the standard contour,

$$\exp(-kz)f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \exp(zs)L(s+k)\,ds \quad (c > 0).$$

Then the contour may be swung back onto the negative real axis if we take account of the poles at s_1 and s_2^+. The contribution from the poles will be

$$C\{g^1(s_1)\exp(s_1 z) - g^1(s_2^+)\exp(s_2^+ z)\}, \quad \text{where } C = -i/\pi\sqrt{(1-k^2)}.$$

Now for $\arg s = \pm\pi$ we have

$$\sqrt{s} = \pm i\sqrt{|s|}, \qquad 1/\sqrt{s} = \mp i/\sqrt{|s|},$$
$$\sqrt{(s+2k)} = \sqrt{|s+2k|} \quad (|s| < 2k),$$
$$\sqrt{(s+2k)} = \pm i\sqrt{|s+2k|} \quad (|s| > 2k).$$

Thus, since $g^1(s) = s^{-1}\exp h(s)$ with $\exp h(s)$ a solution of (2.14),

$$L(s+k) = \frac{i\sqrt{|s+2k|}\exp h(s)}{\pi\sqrt{|s|}\{-i\sqrt{|s^2+2ks|}-i\}},$$

$$L(s^+ + k) = -\frac{i\sqrt{|s+2k|}\exp h(s^+)}{\pi\sqrt{|s|}\{i\sqrt{|s^2+2ks|}-i\}}$$

for $|s| < 2k$, while, for $|s| > 2k$,

$$L(s+k) = \frac{\sqrt{|s+2k|}\exp h(s)}{\pi\sqrt{|s|}\{-\sqrt{|s^2+2ks|}-i\}},$$

$$L(s^+ + k) = \frac{\sqrt{|s+2k|}\exp h(s)\{-\sqrt{|s^2+2ks|}+i\}}{\pi\sqrt{|s|}\{-\sqrt{|s^2+2ks|}-i\}\{-\sqrt{|s^2+2ks|}-i\}}.$$

Thus, after completing the deformation of the path we have

$$\exp(-kz)f(z) = C\{g^1(s_1)\exp(s_1 z) - g^1(s_2^+)\exp(s_2^+ z)\} +$$
$$+\frac{1}{\pi^2}\int\limits_{-2k}^{-\infty} \frac{\sqrt{(2k+s)}\exp\{zs+h(s)\}}{\sqrt{s}\{\sqrt{(s^2+2ks)}+i\}^2}\,ds. \quad (2.20)$$

Now we note that $s_2^+ = s_1^-$ and hence by (2.17) we see that $h(s_2^+) = h(s_1^-)$. One sees then that the first two terms on right of (2.20) combine to yield one term of the form

$$c'\exp(-kz)\cos\{\sqrt{(1-k^2)}(z+\alpha)\},$$

where c' and α are constants. Replacing s in the integral term of (2.20) by $u-2k$ we then obtain the estimate

$$f(z) = c'\cos\{\sqrt{(1-k^2)}(z+\alpha)\} + O(e^{-kz}) \quad (|z| \to \infty;\ \arg z = 0). \quad (2.21)$$

Equation (2.21) has a physical meaning in terms of the wave problem mentioned in the introduction. The solution $f(x)$ of (2.1) is the function $u(x,y)$ on $y=0$, $x>0$. Equation (2.21) then states that the solution obtained by solving (2.1) has the character of a standing wave far away from the edge of the dock. To obtain a solution of the boundary problem with progressive wave character one must introduce a singularity at the edge [see (**2**) and (**3**)].

We wish to verify that the function, $f(z)$, given by equation (2.20), is indeed a solution of (2.1). Observe first that, by its construction, $f(z)$ satisfies the equation

$$2\pi i \exp(-kz) \int\limits_0^z f(\zeta) I_0\{k(\zeta-z)\}\, d\zeta = \Phi(z^-)-\Phi(z), \tag{2.22}$$

$\Phi(z)$ being the inverse transform of the function $g^1(s)$ defined by (2.15) with $g_0(s)=s^{-1}$. Now form the function $F(z)$ defined by (2.3). Then (2.6) holds and hence by (2.22) the difference

$$\exp(-kz)F(z)-\Phi(z)$$

is single-valued. If we can show this difference to be bounded at $z=0$ and to vanish as $|z|\to\infty$ in $-2\pi\leqslant \arg z\leqslant 0$, it follows that

$$F(z)\equiv \exp(kz)\Phi(z).$$

But then, by (2.6) and (2.7),

$$2\pi i \exp(-kz)\int\limits_0^z I_0\{k(\zeta-z\}\left\{f(\zeta)+\pi^{-1}\int\limits_0^\infty K_0\{k|\eta-\zeta|\}f(\eta)\,d\eta\right\}d\zeta$$

$$= \Phi(z^-)-\Phi(z)+i\exp(-kz)\int\limits_0^z I_0\{k(\zeta-z)\}\exp(k\zeta)\{\Phi(\zeta^-)+\Phi(\zeta)\}\,d\zeta. \tag{2.23}$$

On taking the Laplace transform of (2.23) we find the transform of the right-hand side to be

$$g^1(s^+)-\frac{\sqrt{(s^2+2ks)}+i}{\sqrt{(s^2+2ks)}-i}\,g^1(s),$$

and this expression vanishes by (2.14). Hence the right-hand member of (2.23) is zero and so is the left-hand member, which latter fact implies that $f(z)$ is a solution of (2.1).

To estimate $\exp(-kz)F(z)$ we proceed as in (**1**). The arguments of (**1**) show that

$$F(z)\sim Az^{-\frac{1}{2}}\exp(kz) \quad (|z|\to\infty;\ -2\pi<\arg z<0). \tag{2.24}$$

It is clear that the estimate (2.21) for $f(z)$ holds for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$,

and hence the path of integration in (2.3) may be shifted into $\arg s = \pm\epsilon$ for a suitable small positive ϵ. It follows that the estimate (2.4) holds for $-2\pi-\epsilon < \arg z < +\epsilon$ [cf. (1)]. As to $\Phi(z)$ we recall that $g^1(s)$ was constructed so as to be free of singularities in $-\frac{1}{2}\pi < \arg s < \frac{5}{2}\pi$, $s \neq 0$. Further by (2.15) with $g_0(s) = s^{-1}$ it follows that $g^1(s) \to 0$ as $|s| \to \infty$ in this same region. We can write accordingly

$$\Phi(z) = (2\pi i)^{-1} \int\limits_{-i\infty}^{i\infty} \exp(zs)g^1(s)\, ds,$$

the integral running along the imaginary axis. Now we can deform the path of integration in this last integral so that it consists of two rays running from the origin to $\infty \exp i\theta_1$ and $\infty \exp i\theta_2$. Here θ_1 and θ_2 can be chosen by the relations $\theta_1 = \frac{1}{2}\pi - \arg z$ and $\theta_2 = -\frac{1}{2}\pi - \arg z$. Then we have $\arg z = \pm\frac{1}{2}\pi$ on the two rays. Moreover, if

$$-2\pi \leqslant \arg z \leqslant 0,$$

the deformations always keep s in the region $-\frac{1}{2}\pi \leqslant \arg s \leqslant 2\pi$ in which $g(s)$ is analytic and vanishing for $|s| \to \infty$. It follows by the Riemann–Lebesgue lemma that $\Phi(z)$ vanishes as

$$|z| \to \infty, \qquad -2\pi \leqslant \arg z \leqslant 0.$$

3. Sum kernels

We turn next to equation (1.5) of the introduction: that is

$$f(x) = \int\limits_0^\infty (x+\xi)^{-1} \exp\{-(x+\xi)\}f(\xi)\, d\xi. \tag{3.1}$$

Our initial discussion parallels that of § 2. Define for $-\pi \leqslant \arg z \leqslant +\pi$ the function $F(z)$ of $z = x+iy$ by

$$F(z) = \int\limits_0^\infty (z+\xi)^{-1} \exp\{-(z+\xi)\}f(\xi)\, d\xi. \tag{3.2}$$

From the well-known Plemlj formula for the Cauchy integral we find, for $\arg z = 0$,

$$F(ze^{-\pi i}) - F(ze^{+\pi i}) = 2\pi i\lambda f(z), \tag{3.3}$$

while, clearly,

$$\int\limits_0^\infty (z+\xi)^{-1} \exp\{-(z+\xi)\}f(\xi)\, d\xi = F(z) \tag{3.4}$$

again for $\arg z = 0$. We can accordingly rewrite (3.1) as

$$F(ze^{-\pi i}) - F(ze^{+\pi i}) = 2\pi i\lambda F(z). \tag{3.5}$$

Equation (3.5) replaces (2.8) of § 2. We may again use this relation to continue $F(z)$, as defined first by (3.2), onto a logarithmic Riemann

surface. For, with $F(z)$ in $-\pi \leqslant \arg z \leqslant +\pi$, (3.5) determines $F(z)$ in $-2\pi \leqslant \arg z \leqslant -\pi$ and $\pi \leqslant \arg z \leqslant 2\pi$. Upon repeating this procedure indefinitely, we obtain a function $F(z)$ agreeing with the right-hand side of (3.2) on $-\pi \leqslant \arg z \leqslant \pi$ but analytic and satisfying (3.5) on the entire Riemann surface with the origin as logarithmic branch point.

The relation (3.5) differs from those we have considered previously in that it connects functional values of $F(z)$ at three points rather than two. We show now that (3.5) is in fact equivalent to two relations of the type previously considered. To this end introduce

$$\zeta = z^2, \qquad G(\zeta) = F\{z(\zeta)\}.$$

$G(\zeta)$ is then defined by the integral (3.2) for $-2\pi \leqslant \arg\zeta \leqslant +2\pi$. If we write again ζ^+ and ζ^- for image points of ζ on the logarithmic Riemann surface, (3.5) becomes

$$G(\zeta)-2\pi i\lambda G(\zeta^+)-G(\zeta^{++}) = 0, \tag{3.6}$$

where, of course, ζ^{++} denotes $(\zeta^+)^+$.

The equation (3.6) is a 'second-order' difference equation which we reduce to two 'first-order' equations of the type studied in (**1**). We seek constants α, β, γ such that, if

$$g(\zeta) = \alpha G(\zeta)+G(\zeta^+), \tag{3.7}$$

then we have

$$\beta g(\zeta)+\gamma g(\zeta^+) = 0. \tag{3.8}$$

This leads to $\beta\alpha = 1$, $\gamma = -1$, and $\beta+\gamma\alpha = -2\pi\lambda i$, i.e.

$$\beta = -\pi i\lambda \pm \sqrt{(1-\pi^2\lambda^2)} = e^{i\theta_1},\, e^{i\theta_2}. \tag{3.9}$$

Let g_1 and g_2 denote solutions of (3.8) corresponding to θ_1 and θ_2 respectively. Then, by (3.7) and $\alpha = \beta^{-1}$, we have

$$\exp(-i\theta_1)G(\zeta)+G(\zeta^+) = g_1(\zeta), \qquad \exp(-i\theta_2)G(\zeta)+G(\zeta^+) = g_2(\zeta),$$

i.e.

$$G(\zeta) = \{\exp(-i\theta_1)-\exp(-i\theta_2)\}^{-1}\{g_1(\zeta)-g_2(\zeta)\}$$
$$= \tfrac{1}{2}(1-\pi^2\lambda^2)^{-\frac{1}{2}}\{g_1(\zeta)-g_2(\zeta)\}. \tag{3.10}$$

We turn our attention now to the solution of (3.8): that is, by (3.9),

$$g_r(\zeta)-\exp[-i\theta_r]g_r(\zeta^+) = 0 \quad (r = 1, 2). \tag{3.11}$$

Multiplying by $\zeta^{-\theta_1/2\pi}$, $\zeta^{-\theta_2/2\pi}$ respectively, and noting that

$$(\zeta^+)^{-\theta_r/2\pi} = (\zeta e^{2\pi i})^{-\theta_r/2\pi} = \exp(-i\theta_r)\zeta^{-\theta_r/2\pi} \quad (r = 1, 2),$$

we see that (3.11) becomes

$$\{g_r(\zeta)\zeta^{-\theta_r/2\pi}\}-\{g_r(\zeta)\zeta^{-\theta_r/2\pi}\}^+ = 0 \quad (r = 1, 2):$$

that is, the functions

$$\Psi_r(\zeta) = g_r(\zeta)\zeta^{-\theta_r/2\pi} \quad (r = 1, 2) \tag{3.12}$$

are single-valued. For convenience in writing we shall suppress the subscripts 1 and 2 on θ, g, Ψ, it being understood that θ indicates θ_1 and θ_2 and so forth.

In order to determine the function $\Psi(\zeta)$ we must fix its behaviour as $|\zeta| \to \infty$ and $|\zeta| \to 0$. From (3.2) we see [cf. (**1**)] that

$$F(z) = Cz^{-1}\exp(-z) \quad (|z| \to \infty;\ -\pi < \arg z < \pi).$$

Hence

$$G(\zeta) \sim C\zeta^{-\frac{1}{2}}\exp(-\zeta^{\frac{1}{2}}) \quad (|\zeta| \to \infty;\ -2\pi < \arg < 2\pi),$$

and therefore

$$G(\zeta^+) \sim -C\zeta^{-\frac{1}{2}}\exp(\zeta^{\frac{1}{2}}) \quad (|\zeta| \to \infty;\ -2\pi < \arg < 0).$$

Since $\alpha = \exp(-i\theta)$, equation (3.7) then yields the estimates

$$g(\zeta) \sim C\zeta^{-\frac{1}{2}}\{\exp(\zeta^{\frac{1}{2}} - i\theta) - \exp(\zeta^{\frac{1}{2}})\} \quad (|\zeta| \to \infty;\ 2\pi < \arg \zeta < 2\pi).$$

If we rewrite this estimate in the form

$$g_{1,2}(\zeta) \sim C'\zeta^{-\frac{1}{2}}\{\exp(-\zeta^{\frac{1}{2}} - \tfrac{1}{2}i\theta) - \exp(-\zeta^{\frac{1}{2}} + \tfrac{1}{2}i\theta)\} = C''\zeta^{-\frac{1}{2}}\sin(i\zeta^{\frac{1}{2}} - \tfrac{1}{2}i\theta),$$

we recognize the behaviour of $g_1(\zeta)$ for large $|\zeta|$ as that of a Bessel function. In fact, the regular Bessel function $J_{\lambda+\frac{1}{2}}(z)$ has the asymptotic form

$$J_{\lambda+\frac{1}{2}}(z) \sim (2/\pi z)^{\frac{1}{2}}\sin(z - \tfrac{1}{2}\lambda\pi). \tag{3.13}$$

If in particular we set $z = i\zeta^{\frac{1}{2}}$ and $\lambda = \theta/\pi$, we obtain the same behaviour as that desired for g.

The functions $I_\nu(z) = \exp(-\frac{1}{2}\nu i\pi)J_\nu(iz)$ satisfy

$$I_{\lambda+\frac{1}{2}}(z) = z^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty} a_n z^n,$$

and hence the functions

$$\tilde{\Psi}(\zeta) = \zeta^{-\frac{1}{4}-\theta/2\pi}I_{\frac{1}{2}+\theta/\pi}(\zeta^{\frac{1}{2}})$$

are single-valued. Thus we see that the ratios Ψ_1, Ψ_2 being defined by (3.12),

$$\Psi(\zeta) = \Psi(\zeta)/\tilde{\Psi}(\zeta), \tag{3.14}$$

are single-valued functions which are bounded at infinity. The numerators are without poles, and hence these ratios are meromorphic functions with poles only at the zeros of $\tilde{\Psi}(\zeta)$.

Reintroducing the subscripts we have by (3.10) and (3.14),

$$\begin{aligned} F(Z) = G(z^2) &= \tfrac{1}{2}(1-\pi^2\lambda^2)^{-\frac{1}{2}}\{z^{\theta_1/\pi}\tilde{\Psi}_1(z^2)\Psi_1(z^2) - z^{\theta_2/\pi}\tilde{\Psi}_2(z^2)\Psi_2(z^2)\} \\ &= \tfrac{1}{2}z^{-\frac{1}{2}}(1-\pi^2\lambda^2)^{-\frac{1}{2}}\{I_{\frac{1}{2}+\theta_1/\pi}(z)\Psi_1(z^2) - I_{\frac{1}{2}+\theta_2/\pi}(z)\Psi_2(z^2)\}. \end{aligned}$$

We observe by (3.9) that $\theta_2 = -\theta_1 - \pi$; hence $\frac{1}{2} + \theta_2/\pi = -\frac{1}{2} - \theta_1/\pi$.

Also the integral equation (3.1) states that $f(z) = \lambda F(z)$ on $\arg z = 0$, and we obtain for our solution

$$f(z) = \tfrac{1}{2}\lambda z^{-\frac{1}{2}}(1-\pi^2\lambda^2)^{-\frac{1}{2}}\{I_\delta(z)\Psi_1(z) - I_\delta(z)\Psi_2(z^2)\}, \tag{3.15}$$

where $\delta = \frac{1}{2}+\theta_1/\pi$.

The meromorphic functions Ψ_1 and Ψ_2 remain as yet undetermined. The appropriate choice makes itself evident, however, when we attempt to verify that the function defined by (3.15) is really a solution of (3.1). We observe first by (3.13) that both the Bessel functions in (3.15) are $O(e^z/\sqrt{z})$ as $|z| \to \infty$, $\arg z = 0$.

It follows that for any choice of Ψ_1 and Ψ_2, bounded as $|z| \to \infty$, $f(z)$ as defined by (3.15) is such that $e^{-z}f(z) = O(z^{-1})$ as $z \to \infty$. Hence the integral (3.2) exists. Let $F(z)$ be defined by (3.2) for any Ψ_1 and Ψ_2. Then equations (3.3) and (3.4) hold.

Now $I_\lambda(z) = \exp(-\frac{1}{2}i\lambda\pi)J_\lambda(iz)$ and hence, by a known formula [(**9**) 78],

$$I_\delta\{z\exp(\pm i\pi)\} = \exp(\pm i\pi\delta)I_\delta(z),$$
$$I_{-\delta}\{z\exp(\pm i\pi)\} = \exp(\pm i\pi\delta)I_{-\delta}(z). \tag{3.16}$$

Entering (3.16) in (3.15) we find that

$$f(ze^{\pi i}) - f(ze^{-\pi i}) = \{\exp(i\theta_1) - \exp(-i\theta_1)\}f(z) \quad \text{on } \arg z = 0.$$

But $\exp i\theta_1$ is a solution of $\beta-\beta^{-1} = -2\pi i\lambda$; hence we have, finally,

$$f(ze^{\pi i}) - f(ze^{-\pi i}) = -2\pi i\lambda f(z). \tag{3.17}$$

Comparing (3.17) and (3.3) we see that

$$\mathscr{F}(z) = F(z) - f(z)/\lambda$$

is single-valued. Since, as observed before, $e^{-\zeta}f(\zeta) = o(\zeta^{-1})$, it is easy to verify from (3.2) that

$$e^z F(z) \to 0 \quad (|z| \to \infty;\ -\pi \leqslant \arg z \leqslant \pi).$$

If we could conclude that $e^z f(z) \to 0$ as $|z| \to \infty$, then $e^z\mathscr{F}(z)$ would be an entire function vanishing at ∞ and hence would be zero, i.e. we should have $\mathscr{F} = 0$, i.e.

$$f(z) = \lambda F(z)$$

which, for $\arg z = 0$, is equation (3.1). We can achieve the desired effect by choosing $\Psi_1 = \Psi_2$, for in this case we can use the formula [(**9**) 80]

$$K_\lambda(z) = \frac{\pi}{2\sin \pi\lambda}\{I_\lambda(z) - I_{-\lambda}(z)\},$$

where K_λ is the Bessel function of second kind with imaginary argument, to write (3.15) as

$$f(z) = -\pi\lambda z^{-\frac{1}{2}}\frac{(1-\lambda^2\pi^2)^{-\frac{1}{2}}}{4\sin\lambda\pi}K_\delta(z)\Psi_1(z^2).$$

The $K_\delta(z)$ vanish exponentially as $|z| \to \infty$: that is

$$K_\delta(z) = O(e^{-z}/\sqrt{z}) \quad \text{as} \quad |z| \to \infty \quad (-\pi < \arg z < \pi),$$

and hence our verification is complete. The choice $\Psi_1^* = \Psi_2^*$ implies that both are constants since the poles of Ψ_1^*, Ψ_2^* are zeros of $I_{\pm\delta}(z)$ and the latter are distinct. Hence Ψ_1^* can have no poles and, being bounded as $|z| \to \infty$, it must be a constant.

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CLIFFORD GROUPS IN THE PLANE

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In a recent paper in this Journal (**2**), I used properties of Clifford matrices of order 9 to establish the existence in [8] of a four-dimensional locus L, of order 45, invariant under the collineations of a group CT of order 51840×81. As an addendum to that paper it seems worth while to put on record, without proof, some of the results in [2] which arise from the corresponding properties of Clifford matrices of order 3.

Clifford matrices W of order 3 are obtained from those used in my earlier paper by suppressing the second subscripts and superscripts. A *Clifford set* is a set of 3 of these matrices (say W_1, W_2, W_3) such that

$$W_s W_r = \epsilon W_r W_s, \qquad W_3 = \epsilon^t (W_1)^2 W_2$$

($r < s$; $r, s = 1, 2, 3$; $t = 0, 1, 2$; ϵ a complex cube root of unity).

The Clifford groups CG, CS, CT have orders 9, 24, 216 respectively, and CS is the factor group CT/CG. As a symplectic group, CS can be generated by the index matrices

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix} \text{ of period 3, and } \mathbf{Q} = \begin{bmatrix} 1 & 2 \\ 1 & . \end{bmatrix} \text{ of period 6,}$$

with $(\mathbf{QD})^3 = \mathbf{I}$, where the elements of the index matrices belong to $GF(3)$, the Galois field of integers to modulus 3. It is then found that CT is the Hessian group of projective self-collineations of the inflexions of the canonical plane cubic curve. Generators of CT may be taken as

$$D = \operatorname{diag}(1,\ 1,\ \epsilon^2), \qquad Q = \frac{-1}{\sqrt{(-3)}} \begin{bmatrix} 1 & 1 & 1 \\ \epsilon & \epsilon^2 & 1 \\ \epsilon & 1 & \epsilon^2 \end{bmatrix}$$

corresponding to $\mathbf{D}$ and $\mathbf{Q}$. Further, $Q^3 = J$ and $\mathbf{Q}^3 = 2\mathbf{I}$.

Each cubic curve of the pencil

$$x^3 + y^3 + z^3 - kxyz = 0$$

is self-transformed under the 9 involutory collineations JW (x, y, z being homogeneous coordinates in the plane and k a parameter). The JW leave invariant the 9 inflexions and their harmonic polars. All the usual properties of the Jacobian configuration formed by these inflexions can be obtained by means of the algebra outlined above. The

inflexions on the cubic and their harmonic polars, correspond to the solids (on L) and the [4]'s of the configuration previously considered, and the complete Jacobian configuration in [2] corresponds to that in [8].

As a subgroup of CT, CS consists of the elements of CT which commute with one of the nine JW, so that from this point of view there are 9 subgroups CS. Each of these leaves invariant an inflexion and its harmonic polar, and permutes the inflexion triangles in the manner of the alternating group. A typical subgroup CS is generated by WQW^{-1} and WCW^{-1}, where $C = \text{diag}(1, \epsilon, \epsilon)$. Because of the 1–9 correspondence between the index matrices of the symplectic group CS and the matrices of CT (e.g. $\mathbf{Q}^3$ corresponds to the set of nine JW), the symplectic group CS is isomorphic with each of the subgroups CS. Incidentally, as a subgroup of CT, CS is Dickson's group SLH (2, 3) given in (**1**).

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DIAGONALS OF DOUBLY STOCHASTIC MATRICES

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1. A REAL n-square matrix $S = (s_{ij})$ is called *doubly stochastic* if

$$s_{ij} \geqslant 0, \qquad \sum_{k=1}^{n} s_{ik} = \sum_{k=1}^{n} s_{kj} = 1 \tag{1.1}$$

for all i and j. In 1926 van der Waerden ((**4**) 238) conjectured that the minimum value of the permanent of S as S varies over all doubly stochastic matrices is assumed uniquely for the matrix all of whose entries are $1/n$. The *permanent* of S is defined by

$$p(S) = \sum_{\sigma} \prod_{i=1}^{n} s_{i\sigma(i)},$$

where σ varies over all $n!$ permutations of the integers $1,..., n$. To the best of our knowledge this conjecture remains unproved for $n \geqslant 5$ at the present time. As Dr. P. Erdős pointed out to us, the following two statements are consequences of van der Waerden's conjecture:

(a) there exists a permutation σ such that

$$\prod_{i=1}^{n} s_{i\sigma(i)} \geqslant 1/n^n;$$

(b) there exists a permutation σ such that

$$\sum_{i=1}^{n} s_{i\sigma(i)} \geqslant 1$$

and $s_{i\sigma(i)} > 0$ for $i = 1,..., n$.

Of course (b) is an immediate consequence of (a) upon application of the arithmetic-geometric inequality. We have been unable to prove (a); however, in what follows, we do prove a considerably strengthened version of (b) along with some corollaries. Also, we shall obtain uniform lower bounds on the elements of some diagonals of a doubly stochastic matrix that are best-possible.

2. A theorem of Birkhoff's (**1**) states that the set of n-square doubly stochastic matrices is precisely the polyhedron Ω_n obtained by taking the convex hull of the set of n-square permutation matrices. Here by a *permutation matrix* we mean a matrix of the form $(\delta_{i\sigma(i)})$, where σ is

a permutation and δ_{ij} are Kronecker deltas. Thus any doubly stochastic matrix S can be expressed as

$$S = \sum_{j=1}^{m} \theta_j P_j \quad (\theta_j > 0), \qquad \sum_{j=1}^{m} \theta_j = 1,$$

where $P_1,\ldots, P_m$ are permutation matrices. Then

$$S'S = \sum_{j=1}^{m} \theta_j P_j' S;$$

and, if f is any real-valued convex function on Ω_n, we have

$$f(S'S) = f\Big(\sum_{j=1}^{m} \theta_j P_j' S\Big) \leqslant \sum \theta_j f(P_j' S) \leqslant \max_{1 \leqslant j \leqslant m} f(P_j' S). \tag{2.1}$$

As a first result we have

THEOREM 1. *If S is n-square doubly stochastic, $R = S'S = (r_{ij})$, and if $g(t_1,\ldots, t_n)$ is a real-valued function defined for $0 \leqslant t_j < 1$ $(j = 1,\ldots, n)$, and convex with respect to every variable t_j, then there exists a permutation σ such that*

$$g(r_{11},\ldots, r_{nn}) \leqslant g(s_{1\sigma(1)},\ldots, s_{n\sigma(n)}) \tag{2.2}$$

and

$$s_{i\sigma(i)} > 0 \quad (i = 1,\ldots, n). \tag{2.3}$$

Proof. By choosing $f(S) = g(s_{11},\ldots, s_{nn})$ it is clear that (2.2) follows immediately from (2.1). Observe that

$$P_j' S = P_j'\Big(\sum_{i=1}^{m} \theta_i P_i\Big) = \theta_j I + \sum_{i \neq j}^{m} \theta_i P_j' P_i,$$

and hence the elements of the main diagonal of $P_j' S$ are all positive. This establishes (2.3).

By letting $g(t_1,\ldots, t_n) = t_1 + \ldots + t_n$ we obtain from (2.2)

$$\sum_{i,j=1}^{n} s_{ij}^2 \leqslant \sum_{i=1}^{n} s_{i\sigma(i)}, \tag{2.4}$$

which proves (b) since for every i we have

$$\sum_{j=1}^{n} s_{ij}^2 \geqslant 1/n. \tag{2.5}$$

If we let $g(t_1,\ldots, t_n) = t_1$, then (2.2) and (2.5) yield the following interesting corollary:

COROLLARY 1. *If S is n-square doubly stochastic then there exists a permutation σ such that $s_{1\sigma(1)} \geqslant 1/n$ and $s_{i\sigma(i)} > 0$ for $i = 2,\ldots, n$.*

P. Erdős also asked the following question: when can we remove the equality from (2.4)? For this we have only the following partial answer:

COROLLARY 2. *If S is an n-square doubly stochastic matrix all of whose entries are positive, and if $S \neq n^{-1}J$, where J is the matrix all of whose entries are* 1, *then there exists a permutation σ such that*

$$\sum_{i,j=1}^{n} s_{ij}^2 < \sum_{i=1}^{n} s_{i\sigma(i)}.$$

Proof. Let ϵ be positive and sufficiently small and apply (2.4) to $(1-n\epsilon)^{-1}(S-\epsilon J)$. Then we have

$$\sum s_{i\sigma(i)} - n\epsilon \geqslant (1-n\epsilon)^{-1} \sum (s_{ij}-\epsilon)^2,$$

whence

$$\sum s_{i\sigma(i)} \geqslant \sum s_{ij}^2 + n\left(\sum s_{ij}^2 - 1\right)\epsilon + O(\epsilon^2)$$

$$> \sum s_{ij}^2$$

since $S \neq n^{-1}J$ implies $\sum s_{ij}^2 > 1$.

Applying (2.4) to $(n-1)^{-1}(J-S)$, we obtain

COROLLARY 3. *If S is n-square doubly stochastic, then there exists a permutation σ such that*

$$\sum_{i=1}^{n} s_{i\sigma(i)} \leqslant (n-1)^{-1}\left(n - \sum_{i,j=1}^{n} s_{ij}^2\right) < 1.$$

We may generalize the proposition (*b*) as follows (note that any doubly stochastic matrix has 1 as an eigenvalue):

COROLLARY 4. *If S is n-square doubly stochastic and has k eigenvalues of absolute value* 1, *then there exists a permutation σ such that*

$$\sum_{i=1}^{n} s_{i\sigma(i)} \geqslant k, \tag{2.6}$$

and $s_{i\sigma(i)} > 0$ for $i = 1,\ldots, n$. The equality in (2.6) *can be removed unless there exist permutation matrices P and Q such that*

$$PSQ = \sum_{j=1}^{k}{}^{*}\, n_j^{-1} J_{n_j}, \qquad \sum_{j=1}^{k} n_j = n, \tag{2.7}$$

where J_{n_j} is an n_j-square matrix each of whose entries is 1, *and where $\sum^*$ denotes direct sum of matrices.*

For the proof of Corollary 4 we need the following lemma:

LEMMA. *If S is n-square doubly stochastic and has k eigenvalues of absolute value* 1, *then $S'S$ has* 1 *as an eigenvalue of multiplicity at least k.*

Proof. Assume first that S is indecomposable (see (**6**) for a definition). Then by a result of Wielandt's (**6**) there exists a permutation matrix P such that

$$T = P'SP = \begin{pmatrix} 0 & A_1 & 0 & . & . & . & 0 \\ 0 & 0 & A_2 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & A_{k-1} \\ A_k & 0 & 0 & . & . & . & 0 \end{pmatrix}$$

in which in general the A_j are rectangular. But the doubly stochastic property of S implies immediately that each A_j is square. Now choose a permutation matrix R such that

$$TR = \sum_{j=1}^{k}{}^{*} A_j.$$

Then
$$TT' = (TR)(TR)' = \sum_{j=1}^{k}{}^{*} A_j A_j'.$$

Now each $A_j A_j'$ is doubly stochastic and hence has 1 as an eigenvalue. Hence, by noting that $SS' = P(TT')P'$, the argument is completed for S indecomposable since SS' and $S'S$ have the same eigenvalues.

Now, if S is decomposable, we can choose a permutation matrix Q such that

$$QSQ' = \sum_{j=1}^{l}{}^{*} S_j,$$

where each S_j is indecomposable. But

$$(QSQ')(QSQ')' = Q\Big(\sum_{j=1}^{l} S_j S_j'\Big)Q',$$

and we can use the above argument on each S_j to complete the proof.

Proof of Corollary 4. Since $S'S$ is positive semi-definite, we have by the lemma

$$k \leqslant \mathrm{tr}(S'S) = \sum_{i,j}^{n} s_{ij}^2.$$

Hence (2.4) proves the first part of Corollary 4. Now suppose that

$$k = \mathrm{tr}(S'S) = \sum_{i,j}^{n} s_{ij}^2.$$

Then, since $S'S$ has at least k eigenvalues 1, we know that the remaining eigenvalues must be zero. Thus $S'S$ and S have rank k. If we apply (**5**) Theorem 2 [(**5**) 586], (2.7) follows.

We remark that, if we have only that

$$\sup_{\sigma} \sum_{i=1}^{n} s_{i\sigma(i)} = \sum_{i,j=1}^{n} s_{ij}^2,$$

then S will not in general have the form (2.7). For example, let

$$S = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

3. The Frobenius–König theorem (**4**) can be used to obtain the following theorem:

THEOREM 2. *Let S be an n-square doubly stochastic matrix. Then*

(i) *if $n = 2k$, there exists a permutation σ such that*

$$s_{i\sigma(i)} \geqslant \frac{1}{k(k+1)} \quad (i = 1,..., n); \tag{3.1}$$

(ii) *if $n = 2k+1$, then there exists a permutation σ such that*

$$s_{i\sigma(i)} \geqslant \frac{1}{(k+1)^2} \quad (i = 1,..., n). \tag{3.2}$$

These lower bounds are best-possible in the sense that there exist doubly stochastic matrices in every dimension for which the inequality holds for $i = 1,..., n$ and equality holds for some i_0 $(1 \leqslant i_0 \leqslant n)$.

Proof. The proof is by contradiction. Suppose that there exists S of dimension n such that for every σ there exists an i' for which

$$s_{i'\sigma(i')} < \mu.$$

Here μ is $1/k(k+1)$ or $1/(k+1)^2$ according as n is $2k$ or $2k+1$. Now the Frobenius–König result implies that S contains a submatrix M of s rows and t columns $(s+t = n+1)$ such that every element of M is less than μ. Without loss of generality we may assume that S has the form

$$S = \begin{pmatrix} M & B \\ C & D \end{pmatrix},$$

where M, B, C, D are respectively matrices of orders $s\times t$, $s\times(n-t)$, $(n-s)\times t$, $(n-s)\times(n-t)$.

Let m, b, c, d denote respectively the sums of the elements of M, B, C, D. Then

$$m+b = s, \qquad m+c = t,$$

$$2m+b+c = s+t = n+1,$$

$$m+b+c+d = n.$$

Hence
$$m-d = 1, \qquad m \geqslant 1.$$

Also
$$m \leqslant st \max_{\substack{1\leqslant i\leqslant s \\ 1\leqslant j\leqslant t}} s_{ij} < st\mu.$$

Hence
$$\mu > \frac{m}{st} \geqslant 1\Big/\Big(\max_{s+t=n+1} st\Big).$$

Now
$$\max_{s+t=n+1} st = k(k+1) \quad \text{or} \quad (k+1)^2$$

according as $n = 2k$ or $2k+1$, and the proof of (3.1) and (3.2) is complete. The following matrices show that the result is best-possible:

$(n = 2k+1)$

$$\begin{pmatrix} (k+1)^{-2} & . & . & (k+1)^{-2} & (k+1)^{-1} & . & . & (k+1)^{-1} \\ . & . & . & . & . & . & . & . \\ (k+1)^{-2} & . & . & (k+1)^{-2} & (k+1)^{-1} & . & . & (k+1)^{-1} \\ (k+1)^{-1} & . & . & (k+1)^{-1} & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ (k+1)^{-1} & . & . & (k+1)^{-1} & 0 & . & . & 0 \end{pmatrix}.$$

The upper left-hand block is $(k+1)$-square.

$(n = 2k)$

$$\begin{pmatrix} \{k(k+1)\}^{-1} & . & . & \{k(k+1)\}^{-1} & k^{-1} & . & . & k^{-1} \\ . & . & . & . & . & . & . & . \\ \{k(k+1)\}^{-1} & . & . & \{k(k+1)\}^{-1} & k^{-1} & . & . & k^{-1} \\ (k+1)^{-1} & . & . & (k+1)^{-1} & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ (k+1)^{-1} & . & . & (k+1)^{-1} & 0 & . & . & 0 \end{pmatrix}.$$

The upper left-hand block is $k \times (k+1)$.

4. Remark on the theorem of Birkhoff's used in § 2

In (3) Dulmage and Halperin showed that the number m of permutation matrices necessary to express a given n-square doubly stochastic matrix with positive coefficients is at most n^2-n+1. We remark that $m \leqslant (n-1)^2+1$. This is because the convex polyhedron Ω_n formed by all n-square doubly stochastic matrices is, by (1.1), of dimension $(n-1)^2$, and every point in a convex polyhedron of dimension $(n-1)^2$ is contained in the convex hull of m suitable vertices, where $m \leqslant (n-1)^2+1$. The number $(n-1)^2+1$ is of course best-possible.

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THE ANALYSIS OF OBSERVATIONS (III)

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1. Introduction

1.1. The previous papers in this series (**1**), (**4**) have been concerned with the methods used by Eddington in his *Fundamental theory* (**3**). Since the title of the book is a misnomer, it may be as well to make a few remarks on its contents. The book contains no such finished theory but simply a number of arguments with broadly similar principles which purport to formulate and solve a number of physical problems. Amongst the principles formulated, or assumed, in the book are:

(i) physical theories are to be regarded partly as a formulation of conditions determined by our experimental procedures rather than empirically determined;

(ii) a theory may contain explicit reference to its own development in time: such references occur at various points in Eddington (**2**), (**3**). It turns out also that the results which Eddington found could not be found without some such assumption;

(iii) in accordance with a generalization of Mach's principle we must be very careful not to ignore the background in any physical problem;

(iv) a legitimate union of relativity theory and quantum theory consists in solving problems to which both are reasonably good approximations: such problems are difficult to find.

With the use of these principles, and possibly others, Eddington was able to give arguments which resulted in the determination of numerical values for physical constants, at least to his own satisfaction. It is obvious from the four principles stated that such a theory as Eddington's requires careful investigation. Moreover, the present situation in theoretical physics (the difficulties in quantum field theory and the interest which is shown in quantizing the gravitational field) suggests that such an investigation would be valuable.

1.2. In (I) of the present series was set up the necessary algebraic framework for such an investigation, and the analysis was extended in (II), and applied to the particular problem of the fundamental mass-ratio formula

$$m_1/m_2 = k_2/k_1. \tag{1.2.1}$$

The theory of (II), up to and including § 6, will be assumed in the present paper.

In the concluding sections of § 6 certain special assumptions were made in order to deduce Eddington's results, and it is now our opinion that these assumptions are unjustified. The assumptions—that of being separable (Definition 7.2), an integral (Definition 7.5), and especially of being Hamiltonian (Definition 7.7)—correspond to a very special restriction about scale change for which there is no physical justification.

1.3. In the present paper we start from the point reached in (**4**) § 6 and go on to investigate, without special assumptions, whether it is possible to derive the mass-ratio equation. We can then find the conditions under which such a proof can be carried through. These conditions are more general than those of (**4**) and different from those of Eddington.

2. Eddington's argument

2.1. The unusual feature of Eddington's results is the occurrence of numerical values in physical constants, and the most puzzling feature of his book to a beginner is the way in which pure numbers enter at all. It seem probable that all the pure numbers which enter, with the exception of N, do so as multiplicity factors [(**3**) Chapter 2]: that is to say, they occur as the numbers of dimensions in certain phase spaces which are related to measurable quantities, e.g. masses, by the fundamental formula (1.2.1) [(**3**) 16.5]. The approach of Paper II was to make such assumptions that the preparatory ground was clear for the proof of this equation in the same way as by Eddington, but, as we have just noted, these assumptions lack physical content. The possibility of a further investigation of this field is largely due to the publication of Slater's book (**5**). In (**5**) Slater collates a number of separate drafts of 'Fundamental Theory', and his collation enables us to get much nearer to the tangled skein of Eddington's thought than was possible before.

2.2. In particular, we have in (**5**) a number of occurrences of proofs of the fundamental formula (1.2.1). For example, in draft B § 3.3 [(**5**) 111] there is a proof which is followed by the words 'The simplest examples are the transformation of a neutron into a hydrogen atom and of a mesotron into an electron'. At first, this seems to emphasize the importance of Eddington's result since he is clearly regarding it not merely as a result determining a correction factor when we give a different description of a physical system but as a prediction about certain decay

schemes. However, there is clearly something missing, since the decay schemes

$$n \to H^1 + \nu \quad (= p+e+\nu),$$
$$\mu \to e+\nu+\nu,$$
$$\pi \to \mu+\nu,$$

all have as decay products at least two particles. If the fundamental formula is to apply to such a decay scheme, the energy carried away by the neutrino must be very small (a fact which was pointed out to the authors by Mr. H. Bondi). This fact prevents a direct experimental check on the formula (1.2.1), in spite of Eddington's assertion [(**3**) 30] that the mass-ratio is that which would be determined by practical experiment. However, it may well be possible to find some indirect means of comparing the formula with experiment. What is really needed then is an extension of the fundamental formula to cases where there are at least two particles in the end-products. We have not yet found such an extension, but the present paper is a step towards it because it makes quite clear what assumptions are needed for the proof of Eddington's original form.

2.3. Eddington's proof of his fundamental formula rests on two independent ideas. These are

(i) the principles of the rigid field [(**3**) § 13]

and

(ii) the rule for separating energy into field and particle energies [(**3**) §§ 14, 15].

Earlier drafts make Eddington's notion of a rigid field very clear. In draft B 24 III [(**5**) § 6.3] he explains Hartree's self-consistent field thus.

Starting with a prescribed field F, which we treat as rigid, we determine a set of eigenfunctions ψ_r $(r = 1, 2,...)$. Having decided which of these are to be occupied, or more generally having assigned occupation factors j_r to each of the ψ_r, we calculate the field F' which would result from the distribution. We then vary the initial field F until by trial and error we find a 'self-consistent field' for the state of occupation, i.e. a field which satisfies $F' = F$. By the stationary condition, small changes of occupation are admissible without altering F; but any considerable change in j_r involves an entire recalculation of the self-consistent field. Thus the energies H_r of the eigenfunctions are functions of the j_r; and the total energy of the system,

$$H^0 = \sum j_r H_r \tag{3.11}$$

is a non-linear function of the j_r.

In (**3**) the rigid-field convention is finally formulated in the form that quantum mechanics specifies a fixed metrical field, with a frame of eigenstates, and this field must be stationary for small changes in

occupation. It should be clear that this convention is intended by Eddington to be a description of the implicit assumptions of quantum theory, not an account of the explicit method of the theory. As such it is hard to give any example to illustrate the convention. But as it stands it is certainly a consistent method of procedure, and it seems in a general way to describe the practice of quantum mechanics fairly well.

2.4. Now, however, Eddington applies this convention to his rule for separating field and particle energies [(**3**) § 14]. In this rule we consider a system whose total energy H is a general function of the occupation factors $j_1, j_2, \ldots$ corresponding to the (discrete) set of eigenstates ψ_r of the system.

Eddington considers two cases in (**3**) depending upon whether the system is described by discrete or continuous variables. The main interest centres on the continuous case since, as we shall show, a substitute for the rigid-field convention can be found, whereas in the discrete case no such substitute seems possible. The continuous case was also that considered in (**4**). For the sake of simplicity, however, we shall first discuss the discrete case. Under small changes of the occupation factors the corresponding change in the energy is given by

$$\delta H = \sum \frac{\partial H}{\partial j_r} \delta j_r,$$

and this leads us to define

$$E = \sum j_r \frac{\partial H}{\partial j_r}$$

as the total particle energy of the system. Since this total particle energy is different from the original energy, we must compensate by inserting a field energy $W = H - E$.

2.5. The rigid-field convention now states that the allotment must be made in such a way that, under small changes of the occupation factors,

$$\delta W = \delta H - \delta E = 0.$$

Since the field is to be rigid, we have

$$\delta W = \sum \frac{\partial W}{\partial j_r} \delta j_r = 0,$$

so that $\partial W / \partial j_r = 0$. In general W has an extremal value but it is also possible that $W = H - E$ is constant, so that

$$E = \sum j_r \frac{\partial E}{\partial j_r},$$

and in all $$H = E^{(1)}(j_r) + W,$$

where $E^{(1)}(j_r)$ is homogeneous of order 1 in the j_r, and W is constant. This resolution into field and particle energies is valid, but of no interest since the energy was already of the form required.

2.6. It may make the result of this paragraph clearer to give a classical example. Let us consider matter of density ρ moving with velocity $\mathbf{v}$ in an external field of potential $f(\mathbf{r})$, and also under its mutual attraction. The total energy has the form

$$H = \int \{\tfrac{1}{2}\rho\mathbf{v}^2+\rho f(\mathbf{r})\}\,d\tau+\tfrac{1}{2}\iint \rho(\mathbf{r})\rho(\mathbf{r}')\phi(|\mathbf{r}-\mathbf{r}'|)\,d\tau d\tau',$$

where $\phi(|\mathbf{r}-\mathbf{r}'|)$ is the potential function for the mutual action. Then

$$\delta H = \int \delta\rho\{\tfrac{1}{2}\mathbf{v}^2+f(\mathbf{r})\}\,d\tau+\iint \delta\rho(\mathbf{r})\rho(\mathbf{r}')\phi(|\mathbf{r}-\mathbf{r}'|)\,d\tau d\tau',$$

so that

$$\begin{aligned} E &= H+\tfrac{1}{2}\iint \rho(\mathbf{r})\rho(\mathbf{r}')\phi(|\mathbf{r}-\mathbf{r}'|)\,d\tau d\tau' \\ &= H-W. \end{aligned}$$

This gives a division of the total energy H into two parts. It is not profitable at this point to consider the physical significance of this division. If, however, we then apply the rigid-field convention to W, as Eddington requires, we have

$$\iint \delta\rho(\mathbf{r})\rho(\mathbf{r}')\phi(|\mathbf{r}-\mathbf{r}'|)\,d\tau d\tau' = 0,$$

so that

$$\int \rho(\mathbf{r})\phi(|\mathbf{r}-\mathbf{r}'|)\,d\tau = 0$$

for all $\mathbf{r}'$. Thus the field is only rigid when the total energy of the interaction vanishes, and the energy is then all counted as particle energy. This illustrates the result of § 2.5 exactly.

3. The continuous case

3.1. The continuous case has already been treated in (**4**), but we alter some of the notation slightly for the purpose of comparison with (**3**). The state of the system is now described by a point of a phase space whose coordinates are X^α ($\alpha = 1, 2,...$) and instead of the occupation factors for discrete states we have a probability density j, where

$$j(X)\,d\tau = j(X)\,dX^1dX^2\dots dX^k$$

is the probability of finding the system in the volume $d\tau$ at the point X^α. It is a little hard to see what form is chosen by Eddington for the energy in this case. In (**3**) 27 he gives a footnote about Hamiltonian

differentiation referring to the *Mathematical theory of relativity*, where the actual equation quoted is

$$\delta \int G\sqrt{(-g)}\, d\tau = \int \frac{\delta G}{\delta g_{\mu\nu}} \delta g_{\mu\nu}\sqrt{(-g)}\, d\tau,$$

and this suggests that he is considering a total energy of the form

$$H = \int f\, d\tau, \tag{3.1.1}$$

where $$f = f(j, j_{,\mu}, j_{,\mu\nu}, \ldots)$$

[cf. (**4**) Definition 7.5, where F is written for H, and later F_1 for E and F_2 for W]. On the other hand, this form is not actually stated in (**3**), neither does it occur in an early draft [draft C 27 III; (**5**) 88]. For each of these cases the initial formula in the discrete case is not reproduced at all. However, in draft C the formula (21.1) is only true as it stands if H is assumed to be of the form (3.1.1) since otherwise the quantities $H(X)$ which occur will be functionals of the probability distribution. It seems likely that Eddington had not clearly made up his mind whether he would consider the energy to be given in the form (3.1.1) or whether he intended to take the more general form in which the energy is a general functional of the probability distribution.

3.2. Taking the general form $H = H[j]$ for the energy, we find, for a small change in the probability distribution,

$$\delta H = \int \frac{\delta H}{\delta j(X)} \delta j(X)\, d\tau,$$

which leads us to define, in exactly the same way as for the discrete case, the total particle energy E by

$$E = \int \frac{\delta H}{\delta j(X)} j(X)\, d\tau.$$

The field energy will then be

$$W = H - E = H[j] - \int j(X) \frac{\delta H}{\delta j(X)}\, d\tau.$$

The discussion of the rigid-field convention is now a little more complicated, but it can be carried through in much the same way as in the discrete case, and it again leads to a similar result. We have for the variation of W

$$\delta W = \int \frac{\delta W}{\delta j(X)} \delta j(X)\, d\tau,$$

so that $$\frac{\delta W}{\delta j(X)} = 0,$$

and therefore W either has an extremal value, or is a function of X only. In that case

$$\frac{\delta H}{\delta j(X)} = \frac{\delta E}{\delta j(X)},$$

so that

$$E = \int j(X) \frac{\delta E}{\delta j(X)}\, d\tau,$$

which implies that E is a homogenous functional of order 1, i.e.

$$E[\lambda j(X)] = \lambda E[j(X)].$$

This gives, for the functional H, the form

$$H[j] = E^{(1)}[j] + W(X).$$

3.3. Let us then set the rigid-field convention on one side and consider Eddington's derivation of the fundamental formula without it. The essence of this derivation is the use of the scale-free nature of the system which enables us to perform the transformation

$$X^\alpha \to X'^\alpha = (1+\epsilon)X^\alpha.$$

Under this transformation the probability distribution is changed in such a way that

$$j'(X')\, d\tau' = j(X)\, d\tau,$$

(this is the definition of a scale-free system, (**3**) § 9). The first step is to calculate the variation in the distribution. We have

$$j'(X') = j(X)(1-k\epsilon)$$

(when ϵ is small), so that

$$j'(X) = (1-k\epsilon)j(X-\epsilon X),$$

and hence

$$\delta j(X) = -\epsilon(kj(X) + X^\alpha j_{,\alpha}),$$

(with the summation convention). Accordingly the formula for the change of energy becomes

$$\delta H = -\epsilon \int \frac{\delta H}{\delta j}(kj + X^\alpha j_{,\alpha})\, d\tau$$

$$= -k\epsilon E - \epsilon \int X^\alpha j_{,\alpha} \frac{\delta H}{\delta j}\, d\tau.$$

3.4. If we assume the special form (3.1.1) for H, with $f = f(j)$ only, the result for δH simplifies to

$$\delta H = -k\epsilon E - \epsilon \int X^\alpha \frac{\partial f}{\partial X^\alpha}\, d\tau.$$

Then, in Eddington's derivation, the last term becomes

$$-\epsilon \int X^\alpha f_{,\alpha}\, d\tau = \epsilon \int X^\alpha{}_{,\alpha} f\, d\tau = k\epsilon H,$$

so long as f vanishes suitably outside some region; we therefore derive Eddington's result [(**3**) § 15]

$$\delta H = k\epsilon(H-E).$$

For the special form (3.1.1), then, the rigid-field convention is not required, but this special form corresponds to the energy being contributed independently from all the sets: that is to say, it corresponds in the discrete case to the very trivial form in which energy is simply given by the sum

$$H = \sum H_r(j_r).$$

For a general functional there is no immediate reduction of the extra term. It is at this point in (**3**) that the rigid-field convention is used, although in the way in which things are presented there it appears to be used in a slightly different manner.

3.5. The condition on j, H that Eddington's result should be valid is then

$$\int X^\alpha j_{,\alpha} \frac{\delta H}{\delta j(X)}\, d\tau = -kH. \tag{3.5.1}$$

If H has the special form (3.1.1) with f a function of j only, which vanishes outside a certain region, this condition becomes an identity. More generally with f a function of j and $j_{,\alpha}$ we find the condition

$$\int j_{,\alpha} \frac{\partial f}{\partial j_{,\alpha}}\, d\tau = 0,$$

which is satisfied for all j if f is homogeneous of order zero in the $j_{,\alpha}$. We may regard (3.5.1) as a new 'rigid-field convention', but it is of course much more complicated, and lacking in physical interpretation.

If we assume (3.5.1), we are able to give a valid proof of Eddington's results [(**3**) (15.6)] but further problems arise:

(i) It is hard to see any physical significance for the condition in the form (3.5.1).

(ii) It is necessary to investigate the extent to which (3.5.1) is a restriction on the coordinate-system and the extent to which it is a restriction on the physical systems which we consider.

3.6. We here discuss only (ii). Let us first consider a little more closely the nature of the coordinates X^α in the phase space. These coordinates represent the various measurable properties of the system, e.g. energy, momentum, charge, and so on. Thus the geometry of space is of rather a special kind since we are not contemplating transformations of the coordinate-system of the nature of rotations. The allowed

transformations in the space will be, in general, merely those of the form
$$X^\alpha \to \bar{X}^\alpha = \phi^\alpha(X^\alpha), \tag{3.6.1}$$
which correspond to replacing some physical quantity by a function of it. We mention this fact here because the appearance of many of the equations gives one the impression of a covariant theory, but there is no question of covariance here.

If, now, we write
$$\Phi = \int X^\alpha j_{,\alpha} \frac{\delta H}{\delta j(X)}\, d\tau,$$
and we make the transformation (3.6.1) we have
$$d\bar{\tau} = \phi^{1\prime}\phi^{2\prime}\phi^{3\prime}\ldots\phi^{k\prime}\, d\tau = \Pi\, d\tau \quad \text{(say)},$$
where dashes denote differentiation with respect to the various arguments. Since $\bar{j}\, d\bar{\tau} = j\, d\tau$, we have
$$\bar{j} = j/\Pi$$
and
$$\bar{j}_{,\alpha} = \frac{\partial \bar{j}}{\partial \bar{X}^\alpha} = \frac{\partial X^\beta}{\partial \bar{X}^\alpha}\frac{\partial}{\partial X^\beta}(j/\Pi)$$
$$= \frac{1}{\phi^{\alpha\prime}}\frac{\partial}{\partial X^\alpha}(j/\Pi) \quad \text{(unsummed)}$$
$$= \frac{1}{\Pi\phi^{\alpha\prime}} j_{,\alpha} - \frac{\phi^{\alpha\prime\prime}}{(\phi^{\alpha\prime})^2}\frac{1}{\Pi} j.$$
Now $\bar{H} = H$ by definition, so that $\delta\bar{H} = \delta H$ and hence
$$\int \frac{\delta H}{\delta j}\,\delta j\, d\tau = \int \frac{\delta \bar{H}}{\delta \bar{j}}\,\delta \bar{j}\, d\bar{\tau},$$
but
$$\delta j\, d\tau = \delta \bar{j}\, d\bar{\tau},$$
and so
$$\frac{\delta H}{\delta j} = \frac{\delta \bar{H}}{\delta \bar{j}}.$$
In the transformed coordinate-system, then,
$$\bar{\Phi} = \int \bar{X}^\alpha \bar{j}_\alpha \frac{\delta H}{\delta \bar{j}(\bar{X})}\, d\bar{\tau}$$
$$= \sum_\alpha \int \phi^\alpha(X^\alpha)\left\{-\frac{\phi^{\alpha\prime\prime}}{(\phi^{\alpha\prime})^2} j + \frac{1}{\phi^{\alpha\prime}} j_{,\alpha}\right\}\frac{\delta H}{\delta j(X)}\, d\tau.$$
Hence
$$\bar{\Phi} - \Phi = \sum_\alpha \int \left(\frac{\phi^\alpha}{\phi^{\alpha\prime}} - X^\alpha\right) j_{,\alpha}\frac{\delta H}{\delta j(X)}\, d\tau - \sum_\alpha \int \frac{\phi^\alpha \phi^{\alpha\prime\prime}}{(\phi^{\alpha\prime})^2} j(X)\frac{\delta H}{\delta j(X)}\, d\tau.$$
There is obviously a change of form unless $\phi^{\alpha\prime\prime} = 0$. Such a change of form means that, if (3.5.1) is not satisfied in the original coordinate-system, it will not be satisfied in the new one either. Confining our

attention, then, to the case $\phi^{\alpha''} = 0$ we have

$$\phi^{\alpha}(X^{\alpha}) = X^{\alpha}+A^{\alpha},$$

without loss of generality, where the A^{α} are constants. In that case

$$\delta\Phi = \bar{\Phi}-\Phi = \sum_{\alpha} A^{\alpha} \int j_{,\alpha} \frac{\delta H}{\delta j(X)}\, d\tau. \tag{3.6.2}$$

We now show that (3.6.2) is in fact zero. For consider the effect of an infinitesimal change of origin,

$$X^{\alpha} \rightarrow X^{\alpha}+\epsilon^{\alpha},$$

on the expression for H. We have

$$\delta H = \int \frac{\delta H}{\delta j}\, \delta j d\tau = 0,$$

where

$$\delta j(X) = \bar{j}(X)-j(X),$$

$$\bar{j}(X^{\alpha}+\epsilon^{\alpha}) = j(X^{\alpha}).$$

Since

$$\bar{j}(X^{\alpha}) = j(X^{\alpha})-\epsilon^{\alpha} j_{,\alpha},$$

we have

$$\delta H = -\epsilon^{\alpha} \int j_{,\alpha} \frac{\delta H}{\delta j}\, d\tau = 0.$$

Comparing this with (3.6.2) we see that $\delta\Phi = 0$ also. It follows that the restriction $\Phi+kH = 0$ is a restriction on the physical systems considered, not merely on the coordinate-system.

4. Conclusion

4.1. We have shown that Eddington's use of the rigid-field convention combined with the separation of field and particle energies does not provide a proof of his fundamental mass-ratio equation.

We can carry through his derivation if we replace the rigid-field convention by the restriction $\Phi+kH = 0$, which we have shown to be a restriction on the systems considered. The physical interpretation of this restriction is, however, far from clear.

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ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES

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1. CONSIDER the differential equation

$$\frac{d^2\psi}{dx^2}+\{\lambda-q(x)\}\psi = 0, \tag{1.1}$$

where $0 \leqslant x < \infty$, with boundary condition

$$\psi(0)\cos\alpha+\psi'(0)\sin\alpha = 0, \tag{1.2}$$

where α is some fixed real number. Suppose further that $q(x)$ is continuous and tends steadily to infinity as $x \to \infty$. Then the eigenvalue problem defined by (1.1) and (1.2) has a discrete spectrum, with eigenvalues λ_0, λ_1,... and eigenfunctions $\psi_0(x)$, $\psi_1(x)$,... .

For $\lambda > \min q(x)$, define X as the root of the equation $q(x) = \lambda$. If $q(x)$ is a suitably 'smooth' function, and $\sin\alpha = 0$, each of the large eigenvalues is a root of an equation of the form

$$\int\limits_0^X \{\lambda-q(x)\}^{\frac{1}{2}}\,dx = (n+\tfrac{3}{4})\pi+o(1), \tag{1.3}$$

where n runs through positive integers. If $\sin\alpha \neq 0$, each of the large eigenvalues is a root of an equation of the form

$$\int\limits_0^X \{\lambda-q(x)\}^{\frac{1}{2}}\,dx = (n+\tfrac{1}{4})\pi+o(1). \tag{1.4}$$

There are several proofs of this in the literature, the latest being in (**1**), where also the other proofs are surveyed. The proof as given in (**1**) is in two parts. The first establishes (1.3) and (1.4), but leaves in doubt the relation between the integer n which appears on the right-hand side and the suffix m of the eigenvalue λ_m which is the root of the equation. The second part of the proof shows that $m = n$. It involves an argument with Green's functions, but this is more complicated than is necessary for the problem and arose from a premature use of the asymptotic formula for the Bessel functions. It is the aim of this paper to give a simpler proof of the same result, depending on a more detailed

comparison between the distribution of the zeros of the eigenfunctions, and that of the zeros of the Bessel functions to which they approximate.

In § 2 we prove a lemma on Bessel functions which will prove useful in what follows.

In § 3 we consider the formula (1.3), i.e. the case in which the boundary condition is $\psi(0) = 0$. In § 4 we consider the formula (1.4). In § 5 we consider the corresponding problem with interval $(-\infty, \infty)$, $q(x)$ tending to infinity as $x \to \infty$ and as $x \to -\infty$. The formula in this case is

$$\int_{X'}^{X} \{\lambda - q(x)\}^{\frac{1}{2}}\, dx = (n+\tfrac{1}{2})\pi + o(1), \tag{1.5}$$

where X' and X are the roots of the equation $q(x) = \lambda$. Again we show that the eigenvalue concerned is $\lambda = \lambda_n$.

In § 6 we consider the same problem for the equation

$$\frac{d^2\psi}{dr^2} + \left\{\lambda - q(r) - \frac{l^2+l}{r^2}\right\}\psi = 0 \quad (0 < r < \infty), \tag{1.6}$$

where l is a positive integer. This is well known as the radial equation in a three-dimensional problem with spherical symmetry. It was proved in (**2**) that, for any fixed l, each eigenvalue is a root of an equation of the form

$$\int_{0}^{R} \{\lambda - q(r)\}^{\frac{1}{2}}\, dr = (\tfrac{1}{2}l + n + \tfrac{3}{4})\pi + o(1), \tag{1.7}$$

where R is the root of $q(r) = \lambda$. Again we show that it is λ_n which satisfies this equation.

The assumptions on $q(x)$ or $q(r)$ will be:

(*a*) $q(x)$ is three times continuously differentiable;

(*b*) $q(x) \to \infty$ as $x \to \infty$ (and as $x \to -\infty$ in § 5);

(*c*) $q(x)$ and $q'(x)$ are non-decreasing in $(0, \infty)$ (in § 5, $q(x)$ is non-increasing and $q'(x)$ non-decreasing in $(-\infty, 0)$);

(*d*) as $x \to \infty$ (and as $x \to -\infty$ in § 5), $q'(x)/q(x)$, $q''(x)/q'(x)$ and $q'''(x)/q''(x)$ are all $O(1/x)$.

2. Consider the functions

$$u(x) = x^{\frac{1}{2}} J_{\frac{1}{3}}(x), \qquad v(x) = x^{\frac{1}{2}}\{J_{\frac{1}{3}}(x) + J_{-\frac{1}{3}}(x)\}.$$

In the range $x > 0$, each has simple zeros only [Watson (**3**) § 15.21]. As $x \to \infty$,

$$u(x) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(x - \tfrac{5}{12}\pi), \tag{2.1}$$

$$v(x) \sim \left(\frac{6}{\pi}\right)^{\frac{1}{2}} \cos(x-\tfrac{1}{4}\pi), \qquad v'(x) \sim -\left(\frac{6}{\pi}\right)^{\frac{1}{2}} \sin(x-\tfrac{1}{4}\pi). \tag{2.2}$$

It follows that, when n is a large positive integer, $u(x)$ has a zero in the neighbourhood of $x = (n-\frac{1}{12})\pi$, and $v(x)$ has a zero in the neighbourhood of $x = (n-\frac{1}{4})\pi$. This does not determine the number of zeros of $u(x)$ or $v(x)$ in any particular interval $0 < x < \xi$; but it was proved by Watson [(**3**) 497] that, if n is large enough and $\nu > -1$, $J_\nu(x)$ has exactly n zeros in the interval $0 < x < (n+\frac{1}{2}\nu+\frac{1}{4})\pi$; thus $u(x)$ has exactly n zeros in the interval $0 < x < (n+\frac{5}{12})\pi$.

We shall now prove the lemma:

LEMMA. *If n is a sufficiently large integer, $v(x)$ has exactly n zeros in the interval* $0 < x < (n-\frac{1}{12})\pi$.

It follows from (2.1) that the last of the n zeros of $u(x)$ in

$$0 < x < (n+\tfrac{5}{12})\pi$$

occurs approximately at $(n-\frac{1}{12})\pi$. But the positive zeros of any two cylinder functions of the same order interlace [Watson (**3**) 481]. Also the last zero of $v(x)$ in the given range occurs approximately at $(n-\frac{1}{4})\pi$. Hence, if we can show that the first positive zero of $v(x)$ is less than that of $u(x)$, the lemma will be proved.

It follows from Watson (**3**) § 3.12 (2) that

$$u(x)v'(x)-u'(x)v(x) = -\frac{\sqrt{3}}{\pi}. \tag{2.3}$$

Now suppose that the first positive zero of $u(x)$ is at $x = a$, and that the first positive zero of $v(x)$ is greater than a. Both $u(x)$ and $v(x)$ are positive when x is small and positive, and hence $u'(a) \leqslant 0$ and $v(a) > 0$. Hence also

$$u(a)v'(a)-u'(a)v(a) \geqslant 0,$$

which is inconsistent with (2.3). The lemma therefore follows.

Since the zeros of $v(x)$ occur approximately at $(n-\frac{1}{4})\pi$, $(n+\frac{3}{4})\pi$,..., we can actually replace the number $-\frac{1}{12}$ of the lemma by any fixed number between $-\frac{1}{4}$ and $\frac{3}{4}$.

3. We now turn our attention to proving that $\lambda = \lambda_n$ in (1.3). We first observe that, if a solution $\psi(x)$ of (1.1) is an eigenfunction, it has no zeros for $x \geqslant X$; for in this range $\psi(x)$ is convex downwards where it is positive; just to the right of a zero there would be a point where $\psi(x) > 0$, $\psi'(x) > 0$ (or both negative) and so $\psi(x)$ would tend to infinity.

Let
$$z = \int_x^X \{\lambda-q(t)\}^{\frac{1}{2}}\, dt. \tag{3.1}$$

Then as x increases from 0 to X, z decreases steadily from the value

$$Z = \int_0^X \{\lambda - q(t)\}^{\frac{1}{2}}\, dt \tag{3.2}$$

to 0.

It follows from Langer's method, as developed in (**1**), that, for any real λ, there is an L^2 solution $\psi(x)$ of (1.1) such that

$$\psi(x) = \{\lambda - q(x)\}^{-\frac{1}{4}}\eta(x), \tag{3.3}$$

where

$$\eta(x) = v(z) + w(x) \tag{3.4}$$

and

$$|w(x)| < A\lambda^{-\frac{1}{4}}X^{-1} \tag{3.5}$$

(A denotes various positive constants) uniformly over $0 < x < X$. Furthermore

$$\eta'(x) = -\{\lambda - q(x)\}^{\frac{1}{2}}v'(z) + w'(x) \tag{3.6}$$

and

$$|w'(x)| < A\lambda^{-\frac{1}{4}}X^{-1}\{\lambda - q(x)\}^{\frac{1}{2}}. \tag{3.7}$$

If $\psi(x)$ is a constant multiple of an eigenfunction, then $\psi(0) = 0$, $\eta(0) = 0$, and hence

$$v(Z) = -w(0) = O(\lambda^{-\frac{1}{4}}X^{-1}).$$

Hence, by (2.2),

$$\cos(Z - \tfrac{1}{4}\pi) = o(1).$$

Hence there is an integer n such that

$$Z = (n + \tfrac{3}{4})\pi + o(1). \tag{3.8}$$

Consider the ranges of z

$$0 < z < (n - \tfrac{1}{12})\pi, \qquad (n - \tfrac{1}{12})\pi \leqslant z \leqslant Z.$$

In the former, $v(z)$ has exactly n zeros, by the lemma, if λ, and so also n, are large enough. In the latter range, $\cos(z - \frac{1}{4}\pi)$ varies from $\sin(n + \frac{1}{6})\pi$ to approximately $\sin(n+1)\pi$. Hence $v(z)$ has no zero in this range except one in the neighbourhood of $z = Z$. Thus $v(z)$ has exactly n zeros in the range $0 < z < Z$, apart from one in the neighbourhood of $z = Z$. Let them be $z_1, z_2, \dots, z_n$, in increasing order. Let the corresponding values of x be $x_1, x_2, \dots$, in decreasing order.

The function $v(z)$ satisfies the differential equation

$$\frac{d^2v}{dz^2} + \left(1 + \frac{5}{36z^2}\right)v = 0,$$

and so $v(z)$ is concave downwards where it is positive. Let δ be a positive number less than any maximum value of $v(z)$, and let $-\delta$ be greater than any minimum value. In view of (2.2), such a number can be chosen. Let the values of z for which $v(z) = \delta$ be $z_0', z_1', \dots$, with corresponding values $x_0', x_1', \dots$ of x. Let the values of z for which $v(z) = -\delta$

be z_1'', z_2'',... with corresponding values x_1'', x_2'',... of x. Then $v(z) \geqslant \delta$ in the intervals (z_0', z_1'), (z_2', z_3'),...; $v(z) \leqslant -\delta$ in the intervals (z_1'', z_2''),...; and the positive zeros z_1, z_2,... lie in the intervals (z_1', z_1''), (z_2'', z_2'),.... It follows from (2.2) that $|v'(z)|$ has a positive lower bound in those of the intervals (z_i', z_i'') with z_i' sufficiently large; and then, since $v'(z)$ is continuous and not zero in any (z_i', z_i''), that $v'(z)$ also has a positive lower bound in the remaining intervals (z_i', z_i''). Thus there is a positive number B such that $|v'(z)| \geqslant B$ in all intervals (z_i', z_i'').

We can choose λ so large that the right-hand side of (3.5) is less than δ. Then $\eta(x)$ is positive in the intervals (x_i', x_{i-1}') and negative in the intervals (x_i'', x_{i-1}''), and so has a zero in each interval (x_i'', x_i') or (x_i', x_i''). Further, if λ is chosen so large that the right-hand side of (3.7) is less than $B\{\lambda-q(x)\}^{\frac{1}{2}}$, $|\eta'(x)| > 0$ in each of these intervals, and so $\eta(x)$ has just one zero in each interval (x_i'', x_i').

As regards the ends of the interval $(0, X)$, consider first small values of x, i.e. values of z near to Z. Suppose that in the range $(0, Z)$ the greatest of the numbers z_i', z_j'' is z_N', so that $v(z_N') = \delta$ and $|v(z)| < \delta$ for $z_N' < z \leqslant Z$. It follows from (3.6) as before that $|\eta'(x)| > 0$ for $0 \leqslant x \leqslant x_N'$ corresponding to $z_N' \leqslant z \leqslant Z$. Hence $\eta(x)$ has at most one zero in this interval, and so also $\psi(x)$ has at most one zero; it is actually the zero at $x = 0$.

A similar argument shows that $\eta(x)$ has at most one zero in the interval (x_0', X), and this is in fact the zero at $x = X$. Hence $\psi(x)$ has no zero in the interval (x_0', X).

To sum up, therefore, the number of zeros of $\psi(x)$, excluding that at $x = 0$, is the number denoted above by n. However, if $\psi(x)$ is a constant multiple of the eigenfunction $\psi_m(x)$, then it has m zeros apart from that at $x = 0$, and so $m = n$. In view of (3.8), this gives the required result for the formula (1.3).

4. We now turn to the formula (1.4) arising from the more general boundary condition (1.2), where now $\sin\alpha \neq 0$. In terms of $v(z)$ and $w(x)$ this is

$$\{\lambda-q(0)\}^{-\frac{1}{4}}\{v(Z)+w(0)\}\cos\alpha+[\tfrac{1}{4}\{\lambda-q(0)\}^{-\frac{5}{4}}q'(0)\{v(Z)+w(0)\}+$$
$$+\{\lambda-q(0)\}^{-\frac{1}{4}}\{-\{\lambda-q(0)\}^{\frac{1}{2}}v'(Z)+w'(0)\}]\sin\alpha = 0. \quad (4.1)$$

For large λ, the dominant coefficient in this expression is that of $v'(Z)$, and so the condition becomes $v'(Z) = o(1)$. Hence, by (2.2),

$$\sin(Z-\tfrac{1}{4}\pi) = o(1),$$

so that
$$Z = (n+\tfrac{3}{4})\pi+o(1)$$

for some integer n. Hence, by the lemma of § 2 and the remark following it, $v(z)$ has exactly n zeros in the interval $0 < z < Z$, the last of these zeros occurring approximately at $z = Z-\frac{3}{4}\pi$. If we take λ to be an eigenvalue λ_m, then the comparison of the number of zeros of $v(z)$ and $\psi(x)$ goes through as in § 3, except that, with the changed boundary-condition, there is no longer a zero of $v(z)$ in the neighbourhood of $z = Z$, or of $\psi(x)$ at $x = 0$. Hence $\psi(x)$ also has n zeros in the range $(0, X)$. It follows that $m = n$, and we obtain the result stated with regard to (1.4).

Note that, in the case of the general boundary condition (1.2), (11.1) of (**1**) should have an additional term $O(\lambda_n^{-\frac{1}{2}})$ on the right-hand side. This arises from (4.1) above.

5. The interval $(-\infty, \infty)$

In this case $q(x)$ satisfies conditions similar to those assumed above both as $x \to \infty$ and as $x \to -\infty$. Define z and Z as before, and let

$$z' = \int_{X'}^{x} \{\lambda - q(t)\}^{\frac{1}{2}}\,dt, \qquad Z' = \int_{X'}^{0} \{\lambda - q(t)\}^{\frac{1}{2}}\,dt.$$

Then as in (**1**), as $\lambda \to \infty$ through eigenvalues,

$$\cos(Z+Z') = o(1).$$

Hence there is an integer n such that

$$Z+Z' = (n+\tfrac{1}{2})\pi + o(1). \tag{5.1}$$

Let p be the integer such that

$$Z-\pi < (p+\tfrac{1}{4})\pi \leqslant Z,$$

and let $x = x_0$ correspond to $z = (p+\frac{1}{4})\pi$. Then $v(z)$ has p zeros in the interval $0 < z < (p+\frac{1}{4})\pi$, and so $\psi(x)$ has p zeros in $x_0 < x < X$, if λ is large enough. Now

$$Z+Z' = \left(\int_{X'}^{x_0} + \int_{x_0}^{X}\right)\{\lambda - q(t)\}^{\frac{1}{2}}\,dt = Z_0' + Z_0,$$

say, and $Z_0 = (p+\frac{1}{4})\pi$. Hence

$$Z_0' = (n-p+\tfrac{1}{4})\pi + o(1).$$

Hence $v(z')$ has $n-p$ zeros in the interval $0 < z' < Z_0'$, and so $\psi(x)$ has $n-p$ zeros in the interval $X' < x < x_0$. Altogether, if $\psi(x)$ is a constant multiple of $\psi_m(x)$, then $m = n$. The required result for the formula (1.5) therefore follows from (5.1).

6. We now apply similar analysis to the equation (1.6), in which l is supposed fixed. It was shown in (**2**) that (1.7) is equivalent to

$$z(a)+a\surd\lambda = (\tfrac{1}{2}l+n+\tfrac{3}{4})\pi+o(1), \tag{6.1}$$

where
$$z(r) = \int\limits_r^T \left\{\lambda-q(t)-\frac{l^2+l}{t^2}\right\}^{\frac{1}{2}} dt.$$

Here a is a constant, defined in (**2**) as the value of t for which

$$q(t)+(l^2+l)/t^2$$

is a minimum, and T is the zero of the integrand which is greater than a. It is convenient to consider this form of the result here. Thus it is our object to prove that the eigenvalue for which (6.1) is true is in fact λ_n. As in previous cases, this is done by comparing the zeros of true and approximate solutions of (1.6). Now, however, in order to cover the relevant range of values of r it is necessary to consider different approximations in different sections of the range. The analysis of (**2**) shows that there is an L^2 solution $\psi(r)$ of (1.6) such that

$$\psi(r) = Cr^{\frac{1}{2}}J_{l+\frac{1}{2}}(r\surd\lambda)+O(r^{l+3}\lambda^{\frac{1}{2}l+\frac{1}{4}}) \quad (0 \leqslant r \leqslant \lambda^{-\frac{1}{2}}), \tag{6.2}$$

$$\psi(r) = Cr^{\frac{1}{2}}J_{l+\frac{1}{2}}(r\surd\lambda)+O(\lambda^{-\frac{3}{4}}) \quad (\lambda^{-\frac{1}{2}} < r \leqslant b), \tag{6.3}$$

$$\psi(r) = \{\lambda-q(r)-(l^2+l)/r^2\}^{-\frac{1}{4}}\eta(r) \quad (b \leqslant r \leqslant T), \tag{6.4}$$

where $\eta(r) = v(z)+w(r)$, and

$$|w(r)| < A\lambda^{-\frac{1}{2}}T^{-1}$$

uniformly for $b \leqslant r \leqslant T$.

In (6.2), C is a normalization constant. In (6.3), b is any positive constant. These are only trivially different from the results stated in (**2**). There are corresponding formulae for $\psi'(r)$ obtained by formal differentiation with respect to r.

Let p be the integer such that

$$a\surd\lambda-\pi < (p+\tfrac{1}{2}l+\tfrac{1}{2})\pi \leqslant a\surd\lambda,$$

and let $r_0 = (p+\frac{1}{2}l+\frac{1}{2})\pi\lambda^{-\frac{1}{2}}$. Then it follows from the theorem of Watson's already used that, in the range $0 < r < r_0$, $J_{l+\frac{1}{2}}(r\surd\lambda)$ has exactly p zeros. By an argument similar to those used above, this is also true of $\psi(r)$.

It can be verified that (6.1) remains true if we replace a by r_0; thus

$$z(r_0) = (n-p+\tfrac{1}{4})\pi+o(1).$$

Consequently, by the lemma of § 2, $v(z)$ has exactly $n-p$ zeros in the interval $0 < z \leqslant z(r_0)$, and so, as before, $\psi(r)$ has exactly $n-p$ zeros

for $r \geqslant r_0$. Hence $\psi(r)$ has n zeros, excluding that at $r = 0$. If $\psi(r)$ is constant multiple of a the eigenfunction $\psi_m(r)$, it follows that $m = n$.

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